IL
NUOVO CIMENTO
ORGANO DELLA SOCIETÀ ITALIANA DI FISICA
SOTTO GLI AUSPICI DEL CONSIGLIO NAZIONALE DELLE RICERCHE

VOL. XX, N. 2 *Serie decima* 16 Aprile 1961

# A Model for $S$-Wave $\overline{\mathrm{K}}$-$\mathcal{N}$ Scattering.

B. H. BRANSDEN, H. RASHID and R. G. MOORHOUSE
*Department of Natural Philosophy, University of Glasgow - Glasgow*

(ricevuto il 20 Ottobre 1960)

**Summary.** — The importance of pair creation in the scattering of K-mesons by nucleons is investigated by means of a specific model. A model Hamiltonian is employed that allows only the elementary virtual processes $\pi \leftrightarrow \mathcal{N}+\overline{\mathcal{N}}$ and $\overline{\mathrm{K}} \leftrightarrow \mathrm{Y}+\overline{\mathcal{N}}$ where Y stands for a hyperon, $\mathcal{N}$ for a nucleon and $\pi$ for a pion. Three coupled integral equations are obtained for the scattering amplitudes which represent an exact solution to the model problem. These equations are solved numerically and the characteristics of the model are discussed.

## 1. – Introduction.

Although the $\overline{\mathrm{K}}$-meson-nucleon interaction has received considerable theoretical study, the details of the mechanism remain obscure. On the one hand, the application of dispersion relations [1] is rendered uncertain by the complicated structure of the scattering amplitude in the unphysical region and on the other, model calculations starting from an assumed Hamiltonian and employing an approximate calculational method—such as the Tamm-Dancoff—have met with little success. As K-meson-baryon coupling constants appear

[1] K. IGI: *Prog. Theor. Phys.*, **19**, 238 (1958); **20**, 403 (1958); P. T. MATHEWS and A. SALAM: *Phys. Rev.*, **110**, 565, 569 (1958); D. AMATI, E. GALZENATI and B. VITALE: *Nuovo Cimento*, **12**, 627 (1959); M. ISLAM: *Nuovo Cimento*, **13**, 224 (1959).

to be rather weaker than those in the pion-nucleon system, it is not unreasonable to expect the Tamm-Dancoff method to be applicable and indeed for $K^+$-meson-nucleon scattering, a limited success has been obtained [2] for low energy. For $\overline{K}$-meson-nucleon elastic scattering, the position is very different since it has been established that the second order process $\overline{K}+\mathcal{N} \to Y \to \to \overline{K}+\mathcal{N}$ by itself can not account for the very large observed cross-section of about 70 mb at 20 MeV incident energy [3].

A number of authors [4] have emphasized the important effect of the strong absorptive processes on the energy dependence of the $s$-wave $K^-$-p interaction. As DALITZ and TUAN [4] have pointed out, in an interaction like this where absorption processes are strong, the requirements of unitarity impose a close relationship between the various channels possible in the reaction. It is evident therefore that intermediate states involving pions are important in any consideration of $K^-$-p scattering. The calculations by CEOLIN *et al.* [4] taking into accounto the virtual processes shown in Fig. 1 indicate that pion-effects are important but in this calculation they only lead to a comparatively small cross-section with reasonable values of the coupling constant.

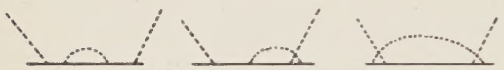

Fig. 1.

This paper has the limited objective of investigating the possible importance of one particular virtual process, namely scattering via the elementary virtual production of pairs,

(1) $\pi \leftrightarrow \mathcal{N}+\overline{\mathcal{N}}'$, and $\overline{K} \leftrightarrow Y+\overline{\mathcal{N}}$.

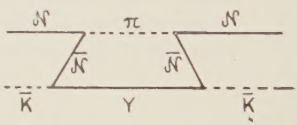


Fig. 2.

This is achieved by using a reduced Hamiltonian for which only the above processes are allowed and hence the only Feynman diagram for $\overline{K}$-$\mathcal{N}$ elastic scattering is that shown in Fig. 2.

The state vector contains only terms in the configurations $(\overline{K}+\mathcal{N})$, $(\mathcal{N}+\overline{\mathcal{N}}+Y)$ and $(\pi+Y)$ and the problem can be solved exactly to obtain the

---

(2) G. BIALKOWSKI and A. JUREWICZ: *Nucl. Phys.*, **17**, 359 (1960).

(3) S. C. FREDEN, F. C. GILBERT and R. S. WHITE: *Phys. Rev.*, **118**, 564 (1960).

(4) J. C. JACKSON, D. G. RAVENHALL and H. W. WYLD Jr.: *Nuovo Cimento*, **9**, 834 (1958); J. C. JACKSON and H. W. WYLD Jr.: *Phys. Rev. Lett.*, **2**, 355 (1959); R. H. DALITZ: in *Proc. of the 1958 Annual International Conference on High Energy Physics at CERN* (Geneva, 1958), p. 187; R. H. DALITZ and S. F. TUAN: *Ann. Phys.*, **8**, 100 (1959); **10** (July 1960); C. CEOLIN, N. DALLAPORTA and L. TAFFARA: *Nuovo Cimento*, **10**, 186 (1958).

$S$-matrix for the reactions

$$\begin{aligned}\overline{\mathrm{K}} + \mathcal{N} &\to \overline{\mathrm{K}} + \mathcal{N}\\ &\to \pi + \Lambda\\ &\to \pi + \Sigma\,.\end{aligned}$$

Another version of the model in which $\pi \rightleftharpoons \mathrm{Y}+\overline{\mathrm{Y}}$ is also allowed is investigated (but clearly higher configurations can now occur and this problem can not be solved exactly).

Apart from its application to $\overline{\mathrm{K}}+\mathcal{N}$ scattering, the model is of interest in itself as it contains three open channels ($\overline{\mathrm{K}}+\mathcal{N}$, $\pi+\Lambda$, $\pi+\Sigma$) and represents a generalization of the model of Bosco and Stroffolini [5] who investigated $s$-wave $\pi$-$\mathcal{N}$ scattering with a model that allowed only the virtual process $\pi \rightleftharpoons \mathcal{N}+\overline{\mathcal{N}}$.

## 2. – Derivation of the integral equation.

From the usual pseudo-scalar interaction Hamiltonian of D'Espagnat and Prentki [6], the terms are retained that lead to virtual processes (1). The reduced interaction is

$$\begin{aligned}(2)\qquad H_I^R = \sum_{r,s,\lambda}\int \mathrm{d}p \int \mathrm{d}q \int \mathrm{d}k\,[a_r^{\mathcal{N}\dagger}(p)\, b_s^{\overline{\mathcal{N}}\dagger}(q)\, C_\lambda^\pi(k)\, \Gamma_\pi(p,q,k) + \text{c.c.} +\\ + b_s^{\overline{\mathcal{N}}\dagger}(q)\, a_r^{\mathrm{Y}\dagger}(p)\, C_\lambda^{\mathrm{K}}(k)\, \Gamma_{\mathrm{K}}(p,q,k) + \text{c.c.}]\,,\end{aligned}$$

where $a_r^{\mathcal{N}}$, $a_r^{\mathrm{Y}}$ are destruction operators for baryons, $b_r^{\overline{\mathcal{N}}}$, $b_r^{\overline{\mathrm{Y}}}$ are those for anti-baryon and $C^\pi$ and $C^{\mathrm{K}}$ are those for $\pi$ and K-mesons respectively. Also $\Gamma_\pi$, $\Gamma_{\mathrm{K}}$ are defined by

$$(3)\qquad \begin{cases} \Gamma_\pi(p,q,k) = \dfrac{G_{\pi\mathcal{N}}}{\sqrt{2\omega_\pi(k)}}\,[\bar{u}_r^{\mathcal{N}}(p) i\gamma_5 \tau_\lambda v_s^{\overline{\mathcal{N}}}(q)]\,\sqrt{\dfrac{M_{\mathcal{N}}^2}{E_{\mathcal{N}}(p)\,E_{\mathcal{N}}(q)}}\,\delta(k-p-q)\,,\\[2ex] \Gamma_{\mathrm{K}}(p,q,k) = \dfrac{G_{\mathrm{Y}\mathrm{K}}}{\sqrt{2\omega_{\mathrm{K}}(k)}}\left[\bar{u}_r^{\mathrm{Y}}(p) i\gamma_5 \begin{Bmatrix}\tau_i\\ 1\end{Bmatrix} v_s^{\overline{\mathcal{N}}}(q)\right]\sqrt{\dfrac{M_{\mathcal{N}}\,M_{\mathrm{Y}}}{E_{\mathcal{N}}(q)\,E_{\mathrm{Y}}(p)}}\,\delta(k-p-q)\,, \end{cases}$$

where

$$\omega_\pi = \sqrt{M_\pi^2 + k^2}\,, \qquad \omega_{\mathrm{K}}(k) = \sqrt{M_{\mathrm{K}}^2 + k^2}\,,$$

$$E_{\mathcal{N}}(p) = \sqrt{M_{\mathcal{N}}^2 + p^2}\,, \qquad E_{\mathrm{Y}}(p) = \sqrt{M_{\mathrm{Y}}^2 + p^2}\,,$$

---

[5] B. Bosco and R. Stroffolini: *Nuovo Cimento*, **10**, 433 (1955).

[6] B. D'Espagnat and J. Prentki: *Nucl. Phys.*, **1**, 33 (1956); *Nuovo Cimento*: **3**, 1045 (1956).

$M_\pi$, $M_{\mathcal{N}}$, etc., being the particle masses and $u$, $v$ are the appropriate positive and negative energy spinors. $G_{\pi\mathcal{N}}$ and $G_{\mathrm{YK}}$ are the unrenormalized coupling constants for the $\mathcal{N}\mathcal{N}\pi$ and $\mathcal{N}\mathrm{YK}$ couplings.

The $\pi$ and $\overline{\mathrm{K}}$-mesons have self-energies through the interactions (1). The state vector for a « real » pion of momentum $k$ can be expressed as

$$(4) \qquad |\Pi_\lambda(k)\rangle = Z_\pi^{\frac{1}{2}}(k)\left[C_\lambda^{\pi\dagger}(k) + \sum_{r,s}\int \mathrm{d}q f_\pi(q)\, a_r^{\mathcal{N}\dagger}(q)\, b_s^{\overline{\mathcal{N}}\dagger}(k-q)\right]|0\rangle ,$$

where $|0\rangle$ is the vacuum state and $Z_\pi$, $f_\pi(q)$ and the self-energy $\delta\omega(k)$ may be found in the usual way by substitution into the Schrödinger equation

$$(5) \qquad H\,|\Pi_\lambda(k)\rangle = \big(\omega(k) + \delta\omega(k)\big)\,|\Pi_\lambda(k)\rangle .$$

The corresponding quantities, $Z_k$, $f_k$ and $\delta\omega(k)$ for the « real » $\overline{\mathrm{K}}$-meson are found in the same way. The resulting integral equations diverge as $k \to \infty$ and so a cut-off is introduced. It should be noted that as recoil effects are included in the calculations, the scattering equations, described below, need no additional cut-off and none has been used.

The state vector for the systems with isotopic spin $T = 0$ and $T = 1$, $\overline{\mathrm{K}}+\mathcal{N}$ and $\mathrm{Y}+\pi$ can be expressed as (in barycentric system)

$$(6) \qquad |\Psi\rangle = \sum_r \int \mathrm{d}k\big[\chi_1(k)\, a_r^{\mathcal{N}\dagger}(-k)\,|\overline{K}(k)\rangle + \chi_2(k)\, a_r^{\Lambda\dagger}(-k)\,|\Pi(k)\rangle +$$

$$+ \chi_3(k)\, a_r^{\Sigma\dagger}(-k)\,|\Pi(k)\rangle\big] + \sum_{r,s,t}\int \mathrm{d}p \int \mathrm{d}q\big[\chi_4(p,q)\, a_r^{\mathcal{N}\dagger}(-q-p)\, b_s^{\overline{\mathcal{N}}\dagger}(q)\, a_t^{\Lambda\dagger}(p)\,|0\rangle +$$

$$+ \chi_5(p,q)\, a_r^{\mathcal{N}\dagger}(-q-p)\, b_s^{\overline{\mathcal{N}}\dagger}(q)\, a_t^{\Sigma\dagger}(p)\,|0\rangle\big] .$$

To obtain equations for the amplitudes $\chi_1$, $\chi_2$, $\chi_3$ for the open channels $\overline{\mathrm{K}}+\mathcal{N}$, $\Lambda+\pi$ and $\Sigma+\pi$ respectively (energies above which the $\mathcal{N}+\overline{\mathcal{N}}+\mathrm{Y}$ channels are also open, are not considered), $|\Psi\rangle$ is inserted into the expression

$$I = \langle\Psi|\,(H - E)\,|\Psi\rangle$$

and the condition $\delta I = 0$ for the variations in the $\chi_i$ imposed. Finally the amplitudes $\chi_4$ and $\chi_5$ can be eliminated and $\chi_1$, $\chi_2$, $\chi_3$ are found to satisfy the coupled equations

$$(7) \qquad [E - E_i(k) - w_i(k)]\, h_i(\boldsymbol{k})\, \chi_i(\boldsymbol{k}) = \sum_{j=1}^{3}\int \mathrm{d}\boldsymbol{k}'\, K_{ij}(\boldsymbol{k}, \boldsymbol{k}')\, \chi_j(\boldsymbol{k}') \qquad i = 1, 2, 3 .$$

The kernels with $i = j$ vanish and the remaining kernels have the same form, for example, $\mathrm{K}_{12}$ is

$$(8) \qquad K_{12}(k, k') = -\frac{C_{12}(T)}{16\pi^3}\sqrt{\frac{M_{\mathcal{N}}^3 M_{\mathcal{N}} M_{\Lambda}}{E_{\mathcal{N}}(k) E_{\Lambda}(k') w(k) w(k') E_{\mathcal{N}}^2(\boldsymbol{k}+\boldsymbol{k}')}}\cdot \\ \cdot\frac{\bar{u}^{\mathcal{N}}(-k)\,\Lambda_{+}^{\mathcal{N}}(k+k')\,u^{\Lambda}(-k')}{E - E_{\mathcal{N}}(\boldsymbol{k}+\boldsymbol{k}') - E_{\Lambda}(k') - E_{\mathcal{N}}(k)}.$$

Here the renormalized coupling constants $g_\pi$ and $g_{\mathrm{K}}$ have been introduced by (*)

$$\bar{G}_\pi^2 = Z_\pi(k) G_\pi^2 = \frac{G_\pi^2}{1 + G_\pi^2 I(k)}, \qquad g_\pi^2 = \lim_{k\to 0} \bar{G}_\pi^2,$$

where

$$(9) \qquad I(k) = \frac{1}{16\pi^3}\int \frac{\mathrm{d}^3 q}{E_{\mathcal{N}}(q) E_{\mathcal{N}}(k+q)\,\omega(k)} \frac{M_{\mathcal{N}}^2 + E_{\mathcal{N}}(k+q) E_{\mathcal{N}}(q) - q\cdot(q+k)}{[E_{\mathcal{N}}(q) + E_{\mathcal{N}}(k+q) - \omega(k)]^2},$$

and similarly $g_{\mathrm{K}}$. The isotopic spin factors $C_{ij}(T)$ including the coupling constants are given in model $B$ by (**)

$$(10) \qquad C(T=1) = \begin{pmatrix} 0 & \sqrt{2}\,\bar{G}_{\Lambda\mathrm{K}}\bar{G}_{\mathcal{N}\pi} & 2\,\bar{G}_{\Sigma\mathrm{K}}\bar{G}_{\mathcal{N}\pi} \\ \sqrt{2}\,\bar{G}_{\Lambda\mathrm{K}}\bar{G}_{\mathcal{N}\pi} & \bar{G}_{\Lambda\Sigma}^2 & -\sqrt{2}\,\bar{G}_{\Lambda\pi}\bar{G}_{\Sigma\pi} \\ \sqrt{2}\,\bar{G}_{\Sigma\mathrm{K}}\bar{G}_{\mathcal{N}\pi} & -\sqrt{2}\,\bar{G}_{\Lambda\pi}\bar{G}_{\Sigma\pi} & -\bar{G}_{\Lambda\pi}^2 + \bar{G}_{\Sigma\pi}^2 \end{pmatrix},$$

and in model $A$, $G_{\Lambda\pi}$ and $G_{\Sigma\pi}$ are put equal to zero. The self-energy term $h_i(x)$ occurring in (7) has the form

$$(11) \qquad h_1(k) = 1 + \frac{G_{\Lambda\mathrm{K}}^2 Z_{\mathrm{K}}}{16\pi^3}\int \frac{\mathrm{d}^3 q}{E_{\mathcal{N}}(k+q) E_{\Lambda}(q) w(k)} \frac{E - E_{\mathcal{N}}(k) - w(k)}{[E_{\mathcal{N}}(k+q) + E_{\Lambda}(q) - w(k)]^2}\cdot \\ \cdot\frac{M_{\mathcal{N}} M_{\Lambda} + E_{\mathcal{N}}(k+q) E_{\mathcal{N}}(q) - q\cdot(k+q)}{E_{\mathcal{N}}(k+q) + E_{\mathcal{N}}(q) + E_{\Lambda}(s) - E},$$

and similarly for channels 2 and 3.

---

(*) It should be noted that the fermion and vertex renormalization factors are both unity in this model.

(**) $$C(T=0) = \begin{pmatrix} 0 & 0 & \sqrt{6}\,\bar{G}_{\Sigma\mathrm{K}}\bar{G}_{\mathcal{N}\pi} \\ 0 & 0 & 0 \\ \sqrt{6}\,\bar{G}_{\Sigma\mathrm{K}}\bar{G}_{\mathcal{N}\pi} & 0 & \bar{G}_{\Lambda\pi}^2 - 2\bar{G}_{\Sigma\pi}^2 \end{pmatrix}.$$

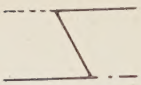

Fig. 3.

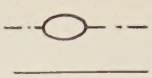

Fig. 4.

In eq. (8) the term corresponds to the process of Fig. 3 whereas the self-energy term eq. (11) corresponds to Fig. 4. The model is constructed so that no vacuum self-energy diagram occurs.

### 3. – Elimination of spin and angular dependence.

At low energies, scattering in states of zero angular momentum is dominant and a harmonic analysis of the kernels to obtain the equations for the state is performed in the standard manner (see for example DYSON *et al.* [7]).

The resulting integral equations are

$$[E - E_i(p) - \omega_i(p)]\, h_i(p)\, a_i(p) = \sum_j C_{ij}(T) \int \frac{s^2\, \mathrm{d}s}{16\pi^2} K_{ij}(p, s)\, a_j(s)\,, \tag{12}$$

where *e.g.*

$$K_{12}(p, s) = \sqrt{\frac{M_{\mathcal{N}}}{M_\Lambda E_{\mathcal{N}}(p) E_\Lambda(s) w(p) \omega(s)}} \cdot$$

$$\cdot \Big\{ [E_{\mathcal{N}}(p) + M_{\mathcal{N}}][X_0^{12}(p, s) + (E_{\mathcal{N}}(p) + E_\Lambda(s) - \Lambda)\, Y_0^{12}(p, s)] +$$

$$+ \frac{ps}{E_\Lambda(s) + M_\Lambda} [X_1^{12}(p, s) + (E_{\mathcal{N}}(p) + E_\Lambda(s) + \Lambda)\, Y_1^{12}(p, s)] \Big\}\,,$$

with

$$X_n^{12}(p, s) = \int_{-1}^{1} \mathrm{d}x\, P_n(x) \frac{1}{E_{\mathcal{N}}(p + s) + E_{\mathcal{N}}(p) + E_\Lambda(s) + E}\,,$$

$$Y_n^{12}(p, s) = \int_{-1}^{1} \mathrm{d}x\, P_n(x) \frac{1}{E_{\mathcal{N}}(p + s) + E_{\mathcal{N}}(p) + E_\Lambda(s) - E} \frac{1}{E_{\mathcal{N}}(p + s)}\,.$$

Although these integrals can be further reduced in the manner of DYSON *et al.*, this was not done but they were instead evaluated directly with an

---

(7) F. J. DYSON, M. ROSS, E. E. SALPETER, S. S. SCHWEBER, M. K. SUNDARESAN, W. M. VISSCHER and H. A. BETHE: *Phys. Rev.*, **95**, 1644 (1945).

electronic computer. Since we have confined ourselves to energies below the threshold for the process $\overline{\mathrm{K}}+\mathcal{N}\to \mathrm{Y}+\mathcal{N}+\overline{\mathcal{N}}$, the singularity in the above integrals does not need to be considered.

## 4. – Non-singular amplitudes and $S$-matrix.

For numerical work, it is convenient to work with real solutions of (12) that correspond to standing waves in configuration space,

$$r\Phi_i(r)\sim \frac{A_i}{k_i}\sin k_i r + B_i \cos k_i r\,. \tag{13}$$

Three independent solutions may be found, with amplitudes $A_i^\mu$, $B_i^\mu$ ($\mu = 1, 2, 3$) and the reaction matrix may then be determined from the equations

$$\sqrt{v_i}\,B_i^\mu = \sum_j R_{ij} A_i^\mu \sqrt{v_j}\,, \tag{14}$$

where $v_i$ is the relative velocity in channel $i$.

Also the $S$-matrix can be obtained from the relation

$$S = \frac{1+iR}{1-iR}\,. \tag{15}$$

The boundary condition (14) is introduced in momentum space by writing

$$a_i(k) = \lambda_i \delta(E - E_i(k) - \omega_i(k)) + P\frac{1}{E - E_i(k) - \omega_i(k)} f_i(k)\,, \tag{16}$$

giving coupled integral equations for the real non-singular amplitudes $f_i(k)$. The $\lambda_i$ are arbitrary parameters, three different sets of $\lambda_i - \lambda_i^\mu$ ($\mu = 1, 2, 3$) determine three independent sets of solutions $f_i^\mu(k)$ allowing the equations (15) to be formed as

$$\begin{cases} A_i^\mu \sim \lambda_i^\mu E_i(k_i)\,\omega_i(k_i)/E \\ B_i^\mu \sim -\pi f_i^\mu(E, k_i)\,E_i(k_i)\,\omega_i(k_i)/E \end{cases} \tag{17}$$

and thus the $R$-matrix and hence the $S$-matrix can be found.

In the second version of the model where the additional interactions

$$\pi \rightleftharpoons \Sigma + \overline{\Lambda},\ \Sigma + \overline{\Sigma},\ \Lambda + \overline{\Sigma}$$

are allowed, the kernels $K_{22}$, $K_{33}$ are no longer zero but take a form similar to $K_{12}$ above. In this case, the solution is no longer an exact solution of the model problem.

## 5. – Numerical calculations.

The equations to be solved numerically are from (12) and (16) of the form

$$f_i^{\mu}(k) = \sum_j \left[ \lambda_i^{\mu} C_{ij}(k) + P \int_0^{\infty} \frac{K_{ij}(k, k') f_j^{\mu}(k')}{E - E_j(k') - \omega_j(k')} \right] k'^2 \, \mathrm{d}k' ,$$

where

$$C_{ij}(k) = \int_0^{\infty} \delta(E - E_j(k) - \omega_j(k)) \, K_{ij}(k, k') \, k'^2 \, \mathrm{d}k' . \tag{18}$$

These equations were expressed as difference equations using 72 pivotal points and gaussian integration formulae for the integrals. Special care has to be taken to remove the singularity in the integrand and this can be achieved by a suitable transformation of the equations, introduced by GAMMEL [8]. Since we have used Gaussian integration, no additional continuity condition on the transformed kernels was required. All the operations and calculations of $S_{ij}$ can be put into matrix form and are particularly suitable for high speed computation on an electronic computer.

The cross-sections are calculated from the expression

$$\sigma_{ij} = \frac{4\pi}{p_i} |T_{ij}|^2 \, p_j ,$$

where

$$S_{ij} = \delta_{ij} + 2i \sqrt{p_i} \, T_{ij} \sqrt{p_j} .$$

## 6. – Results.

To reduce the number of coupling constants $3G^2_{\mathcal{N}\mathrm{K}\Sigma}/4\pi$ was made equal to $G^2_{\mathcal{N}\mathrm{K}\Lambda}/4\pi = G^2/4\pi$ and in the second version of the model (model $B$), the pion coupling constants were made equal $G_{\mathcal{N}\mathcal{N}\pi} = G_{\Sigma\Sigma\pi} = G_{\Lambda\Sigma\pi}$. It should be noted that in common with the Lee model, and several other models, while

---

[8] J. L. GAMMEL: *Phys. Rev.*, **95**, 209 (1954).

the unrenormalized coupling constants may take any real value, there is only a limited range in which the renormalized coupling constants must lie if the theory is to remain hermitian and ghost states are to be avoided.

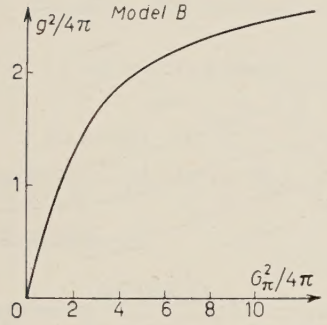


Fig. 5. – The renormalized coupling constant $g^2/4\pi$ as a function of the unrenormalized constant $G_\pi^2/4\pi$.

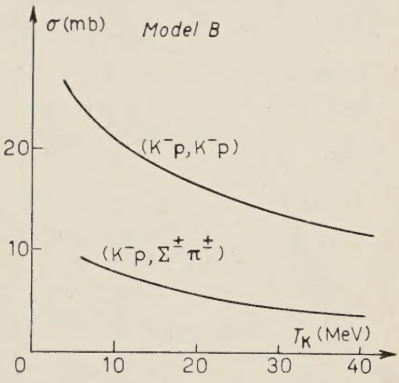


Fig. 6. – The energy variation of the cross-sections for elastic scattering of $K^-$-mesons on protons and the production of $\Sigma^\pm$ hyperons by $K^-$-mesons on protons for $G_\pi^2/4\pi = G_K^2/4\pi = 5.0$.

This follows from the definition eq. (9),

$$G_i^2 = \frac{g_i^2}{1 - g_i^2 I},$$

and we must have $g_i^2 I < 1$ in order that $G_i^2 > 0$. The relation between the renormalized and the unrenormalized coupling constants depends on the value of the cut-off chosen when evaluating the integral $I$. In the present work, two cut-offs $k_{max} = M_\Lambda$ and $0.5\ M_\Lambda$ were initially tried and in Fig. 5, one example of the relationship between $g$ and $G$ is shown for a cut-off $0.5\ M_\Lambda$. In the remainder of the figures, for convenience, the cross-sections are displayed as a function of the unrenormalized coupling constants $G$ for this cut-off.

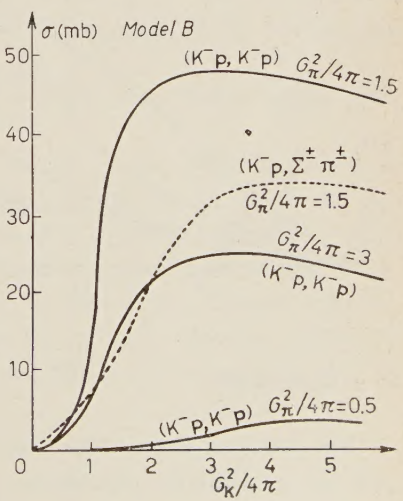


Fig. 7. – The variation on model $B$ of the elastic cross-section with unrenormalized coupling constant $G_K^2/4\pi$ for various values of $G_\pi^2/4\pi$. The production cross-section for $\Sigma^\pm$ hyperons is also shown for one value of $G_\pi^2/4\pi$.

The energy variation of the elastic and inelastic cross-sections for $G_\pi^2/4\pi = G_K^2/4\pi = 5$ is shown for model $B$ in Fig. 6 for a laboratory $K^-$-meson energy of up to 40 MeV. The remainder of the Figs. 7 to 9 display the variation of the cross-section with $G_\pi$ and $G_K$ at a single energy of 20 MeV for both models $A$ and $B$. For the displayed results $G_\pi$ and $G_K$ were taken to be of the same sign. The distinct case where $G_\pi$ and $G_K$ are of opposite sign has been investigated and the results in Fig. 10 are found to follow the same pattern.

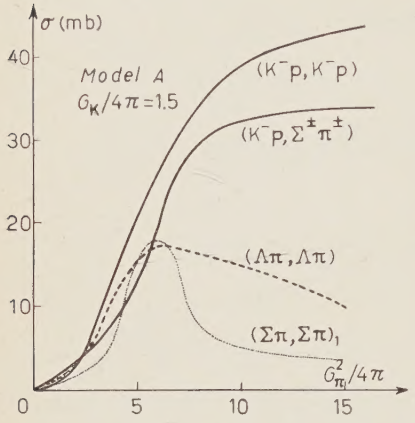


Fig. 8. – The variation on model $A$ of different elastic and production cross-sections for the coupling constant $G_K^2/4\pi = 1.5$.

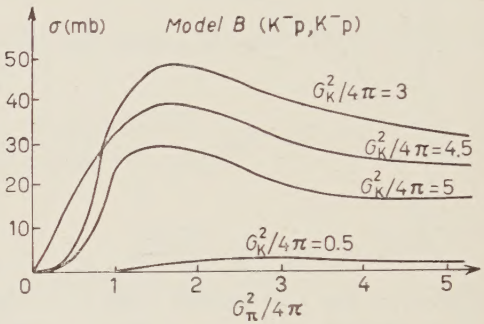


Fig. 9. – The variation on model $B$ of elastic cross-sections against pion coupling constant $G_\pi^2/4\pi$ for various values of $G_K^2/4\pi$.

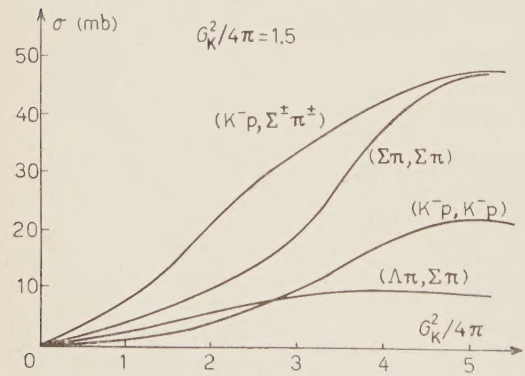


Fig. 10. – The variation of cross-sections for elastic and production processes against $G_K^2/4\pi$ in the case when $G_K$ is negative relative to $G_\pi$.

The amount of computing machine time required for each set of coupling constants turned out to be roughly eight hours and no detailed energy variation for various coupling constants was attempted.

## 7. – Conclusions.

The reduced Hamiltonian we have discussed here is a model which can only provide qualitative indications of a certain aspect of the interaction mecha-

nism namely virtual pair creation in the intermediate states. Because a reduced model Hamiltonian has been used, the coupling constants $G_\pi$ and $G_\mathrm{K}$ are not directly comparable with those employed in calculations using the full Hamiltonian. However, it is apparent that for reasonable values of $G_\pi$ and $G_\mathrm{K}$, a substantial fraction of the observed cross-sections can be obtained with this process. For example with

$$G_\pi^2/4\pi = 1.5\,, \qquad G_\mathrm{K}^2/4\pi = 3\,,$$

which corresponds to the renormalized coupling constants

$$g_\pi^2/4\pi \sim 1.0\,, \qquad g_\mathrm{K}^2/4\pi \sim 2.0\,,$$

the cross-section $\sigma_\mathrm{el}$ turns out to be about 50 mb at 20 MeV in model $B$. Although quantitative agreement with experimental data is not of particular importance in such model calculations, it is interesting to observe that the values of coupling constant $g_\mathrm{K}^2/4\pi$ obtained from dispersion relations ([1]) is of the order of 4. The value of $g_\pi^2/4\pi \sim 1.0$ should also be compared with the result of Bosco and Stroffolini ([3]) who found that a coupling constant $g_{\mathcal{N}\mathcal{N}\pi}^2/4\pi$ of the order of unity reproduced the $S$-wave pion-nucleon phase-shift behaviour reasonably well.

As regards the sign of the $\mathrm{K}^-$-p interaction potential, the convention ([4]) we use is that a positive sign of the real part of the $T$-matrix corresponds to an attractive potential. It has been found that in model $A$, Re $T_{11}$ which corresponds to elastic $\overline{\mathrm{K}}$-$\mathcal{N}$ scattering is positive for the coupling constant $G_\pi^2/4\pi$ varying between 1 and 5 and for both isotopic spins. In the model $B$, Re $T_{11}$ is positive for the coupling constants $G_\pi^2/4\pi$ and $G_\mathrm{K}^2/4\pi$ varying up to 1.5 and for both isotopic spins.

The occurrence of the same sign in both isotopic spin states agrees with the solutions of Dalitz and Tuan.

From Figs. 7, 8 and 9 it can be seen that the cross-section increases in the way demanded by low order perturbation theory for small $G_\pi$ or $G_\mathrm{K}$. However for large values of coupling, the coupling between the three channels becomes important and this results in a definite maximum in the cross-section as a function of $G$. This phenomenon is known in other connections, for example the coupled equations of the type occurring in electron scattering by atoms have been investigated by Massey and Mohr with somewhat similar results.

* * *

We thank Professor J. C. GUNN for his interest in the work and the Computing Laboratory of the University of Glasgow for the use of the DEUCE.

---

## RIASSUNTO (*)

A mezzo di un modello speciale studiamo l'importanza della creazione di coppie nello scattering dei mesoni K sui nucleoni. Usiamo un modello di Hamiltoniano che consente solo i processi virtuali elementari $\pi \leftrightarrow \mathcal{N} + \mathcal{N}$ e $\overline{K} \leftrightarrow Y + \mathcal{N}$, in cui Y è un iperone, $\mathcal{N}$ un nucleone e $\pi$ un pione. Otteniamo per le ampiezze di scattering tre equazioni integrali accoppiate che rappresentano una soluzione esatta del problema del modello. Risolviamo numericamente queste equazioni e discutiamo le caratteristiche del modello.

---

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. XX, N. 2 16 Aprile 1961

# The Determination of the Sign of the Asymmetry Parameter in $\Lambda^0$ Decay from the Scattering of the Decay Proton (*).

T. Bowen, C. R. Sun and A. E. Werbrouck (**)

*Princeton University - Princeton*

(ricevuto il 5 Dicembre 1960)

**Summary.** — A statistical method is presented whereby the information provided by a small number of nuclear scatters of $\Lambda^0$ decay protons can be used to determine the sign of the asymmetry parameter due to parity non-conservation in $\Lambda^0$ decay. As an example, the analysis is applied to twenty-one proton scatters of $\Lambda^0$ hyperons decaying in the Princeton multiplate cloud chamber. Because the asymmetry to be expected in each observed nuclear scatter is small, our data do not yield a statistically significant result. However, if each experimenter would follow a similar procedure with his data, it should be possible to combine results to determine the sign with a high degree of certainty.

## 1. – Introduction.

Recently, Birge and Fowler [1] have reported observations in a propane bubble chamber which indicate a positive helicity for the proton from $\Lambda^0$ decay, in disagreement with a previous cloud chamber measurement [2,3] and the

(*) Supported by the joint program of the Office of Naval Research and The U.S. Atomic Energy Commission.

(**) Now at the Istituto di Fisica, Torino, Italia.

(1) R. W. Birge and W. B. Fowler: *Phys. Rev. Lett.*, **5**, 254 (1960).

(2) E. Boldt, H. S. Bridge, D. O. Caldwell and Y. Pal: *Phys. Rev. Lett.*, **1**, 256 (1958).

(3) E. Boldt: *Ph. D. Thesis, M.I.T.* (1958) (unpublished).

predictions based upon a universal Fermi interaction [4]. In an experiment of the Princeton group with a multiplate cloud chamber with C, Fe, and Pb plates [5], 189 $\Lambda^0$ decays were studied. The scatterings of the decay protons were measured, and statistical methods were devised to deal with such data, in the hope of gaining significant information about the sign of the helicity of the proton in $\Lambda^0$ decay [6]. However, the number of events in our cloud chamber was too small. In view of the disagreement between the experimental results now available, and the theoretical importance of the question, it seems likely that further experiments will be carried out to determine the proton helicity in $\Lambda^0$ decay. Perhaps it is timely, therefore, to describe the statistical methods we used, and to encourage each experimenter to calculate and publish certain statistical quantities which may be combined with the results of others to determine the best estimate and the level of confidence for the sign of the helicity.

It has been observed that $\Lambda^0$'s produced in $\pi^-$-proton collisions exhibit an up-down asymmetry in decay, which demonstrates not only that polarized $\Lambda^0$'s are produced, but that parity is violated in the decay process [7,8]. The angular distribution of the decay protons in the center of mass system of the $\Lambda^0$ can be written

$$\frac{\mathrm{d}N}{\mathrm{d}\Omega} = \frac{1}{4\pi}\,(1 - \alpha P \cos \Theta_{\mathrm{p}}^*)\,, \tag{1}$$

where $\alpha$ is the asymmetry parameter of the decay process as defined by LEE and YANG [9], $P$ is the polarization of the $\Lambda^0$, and $\Theta^*$ is the c.m.s. angle between the axis of polarization of the $\Lambda^0$ and the decay proton. From the magnitude of the up-down asymmetry, it is possible to give a lower limit for the absolute value of $\alpha$, but its algebraic sign is not determined, since the

[4] R. P. FEYNMAN and M. GELL-MANN: *Phys. Rev.*, **109**, 193 (1958); E. SUDARSHAN and R. MARSHAK: *Phys. Rev.*, **109**, 1860 (1958); J. J. SAKURAI: *Nuovo Cimento*, **7**, 649 (1958).

[5] T. BOWEN, J. HARDY, Jr., G. T. REYNOLDS, C. R. SUN, G. TAGLIAFERRI, A. E. WERBROUCK and W. H. MOORE: *Phys. Rev.*, **119**, 2030 (1960).

[6] The data presented in the present paper are discussed in somewhat more detail by T. BOWEN, C. R. SUN and A. E. WERBROUCK: Technical Report no. 21, *Elementary Particles Laboratory* (Princeton, 1958) (unpublished).

[7] F. S. CRAWFORD, Jr., M. CRESTI, M. L. GOOD, K. GOTTSTEIN, E. M. LYMAN, F. T. SOLMITZ, M. L. STEVENSON and H. K. TICHO: *Phys. Rev.*, **108**, 1102 (1957).

[8] F. EISLER, R. PLANO, A. PRODELL, N. SAMIOS, M. SCHWARTZ, J. STEINBERGER, P. BASSI, V. BORELLI, G. PUPPI, G. TANAKA, P. WOLOSCHEK, V. ZOBOLI, M. CONVERSI, P. FRANZINI, I. MANNELLI, R. SANTANGELO, V. SILVESTRINI, D. A. GLASER, C. GRAVES and M. L. PERL: *Phys. Rev.*, **108**, 1353 (1957).

[9] T. D. LEE and C. N. YANG: *Phys. Rev.*, **108**, 1645 (1957); **109**, 1755 (1958).

direction of the polarization is not known. The limits reported for $|\alpha|$ are

$$(2) \qquad \begin{cases} |\alpha| \geqslant .67 \pm .13 \quad ([8]), \\ |\alpha| \geqslant .73 \pm .14 \quad ([11]). \end{cases}$$

It has been pointed out that parity violation in the $\Lambda^0$ decay process implies that the decay proton in the c.m.s. must have a component of polarization along its direction of motion, even if the $\Lambda^0$ is unpolarized. It can be shown ([9,10]) that, in the case of an unpolarized $\Lambda^0$, the transverse polarization of the proton in the $\Lambda^0$ c.m.s. is zero and the longitudinal polarization is $-\alpha$. The Lorentz transformation to the laboratory system changes, in general, both the direction of motion and the spin direction, but in different ways, thus introducing a transverse component of polarization. Non-relativistically, the spin direction remains the same in the c.m.s. and the laboratory system, and, for the events observed in the Princeton cloud chamber, this is a satisfactory approximation ([3]). If $\Theta_p$ is the laboratory angle between the $\Lambda^0$ line of flight and the direction of motion of the proton, the tranverse polarization, $\sigma_T$, along $(\boldsymbol{P}_\Lambda \times \boldsymbol{P}_p) \times \boldsymbol{P}_p$ is given by

$$(3) \qquad \sigma_T = -\alpha \sin (\Theta_p^* - \Theta_p) .$$

## 2. – Nuclear scattering as a polarization analyser.

Extensive experimental studies have been made ([12,13]) of the polarizing strength of elastic nuclear scatters. The results of this work give, for an unpolarized beam of protons of momentum $p_i$, incident on an unpolarized target nucleus of atomic weight $A$, an analyzer strength, $S_\Theta(A, p_i)$, which is the polarization of the protons scattered by an angle $\Theta$. $S_\Theta$ is almost always positive with respect to the scattering plane normal, $\boldsymbol{p}_i \times \boldsymbol{p}_f$, where $\boldsymbol{p}_i$ is the proton momentum before scattering, and $\boldsymbol{p}_f$ the momentum after scattering. This polarization must be transverse, since parity is conserved in the strong interactions producing the scatter. In the inverse situation, where a polarized

([10]) J. Leitner: *Nuovo Cimento*, **8**, 68 (1958).

([11]) F. S. Crawford, Jr., M. Cresti, M. L. Good, K. Gottstein, E. M. Lyman, F. T. Solmitz, M. L. Stevenson and H. K. Ticho: *CERN Conference* (1958).

([12]) K. Strauch and F. Titus: *Phys. Rev.*, **103**, 200 (1956); J. M. Dickson and D. C. Slater: *Nuovo Cimento*, **6**, 235 (1957); J. M. Dickson, B. Rose and D. C. Slater: *Proc. Phys. Soc. (London)*, A **68**, 361 (1955); A. Alphonce, A. Johansson and G. Tibell: *Nucl. Phys.*, **4**, 643 (1957).

([13]) O. Chamberlin, E. Segrè, R. D. Tripp, C. Wiegand and T. Ypsilantis: *Phys. Rev.*, **102**, 169 (1956).

beam of protons is incident on unpolarized target nuclei, let $\varphi$ be the angle between the direction of transverse polarization, $(\boldsymbol{p}_\Lambda \times \boldsymbol{p}_p) \times \boldsymbol{p}_p$, and the scattering plane normal, $\boldsymbol{p}_i \times \boldsymbol{p}_f$, where $\boldsymbol{p}_\Lambda$ is the $\Lambda^0$ momentum and $\boldsymbol{p}_p$ the decay proton momentum in the laboratory. The distribution of $\varphi$ is then given by

$$\left(\frac{\mathrm{d}N}{\mathrm{d}\varphi}\right)_{\text{elastic}} = \frac{1}{2\pi}\left[1 + \sigma_T S_\Theta(A, p_i)\cos\varphi\right]. \tag{4}$$

In general, it is not possible in any one event to distinguish a nuclear scattering, which acts as an analyzer of polarization, from a multiple Coulomb scattering or an inelastic nuclear scattering which is negligibly affected by the proton polarization. Let a proton of a fixed momentum traverse a plate. The probability of a nuclear scattering by an angle $\Theta$ into an element of solid angle $\mathrm{d}\Omega$ is $(\mathrm{d}N/\mathrm{d}\Omega)_{\text{elastic}} \cdot \mathrm{d}\Omega$. The probability of a multiple Coulomb or inelastic scatter into the same element of solid angle at $\Theta$ is $[(\mathrm{d}N/\mathrm{d}\Omega)_{\text{Coulomb}} + (\mathrm{d}N/\mathrm{d}\Omega)_{\text{inelastic}}] \cdot \mathrm{d}\Omega$. The probability, $\eta_\Theta$, that a scattering by an angle $\Theta$ is a nuclear scattering is

$$\eta_\Theta(A, p_i, x) = \frac{(\mathrm{d}N/\mathrm{d}\Omega)_{\text{elastic}}}{(\mathrm{d}N/\mathrm{d}\Omega)_{\text{elastic}} + (\mathrm{d}N/\mathrm{d}\Omega)_{\text{Coulomb}} + (\mathrm{d}N/\mathrm{d}\Omega)_{\text{inelastic}}}, \tag{5}$$

where $\eta_\Theta$ is a function of the scattering angle, $\Theta$; the identity of the target nuclei, $A$; the proton momentum before scattering, $p_i$; and the thickness of material, $x$.

For the analysis of our cloud chamber events, the nuclear scattering distribution was based upon the Born approximation formula ([13]) for the diffraction scattering cross-section, with the constants adjusted to give a reasonable fit to experimental data which have been reported ([12,13]). Curves for the Coulomb scattering were obtained by differentiating the integral distribution given by OLBERT ([14]). The contribution of $(\mathrm{d}N/\mathrm{d}\Omega)_{\text{inelastic}}$ was neglected.

For each scattering which is observed, there is a probability, $\eta_\Theta$, that it is a nuclear scatter, in which case the distribution over $\varphi$ goes according to eq. (4). There is a probability $(1 - \eta_\Theta)$ that the event is a multiple Coulomb or inelastic scatter, in which case the distribution over $\varphi$ is

$$\left(\frac{\mathrm{d}N}{\mathrm{d}\varphi}\right)_{\text{Coulomb or inelastic}} = \frac{1}{2\pi}. \tag{6}$$

The combined distribution is

$$\frac{\mathrm{d}N}{\mathrm{d}\varphi} = \eta_\Theta\left(\frac{\mathrm{d}N}{\mathrm{d}\varphi}\right)_{\text{nuclear}} + (1 - \eta_\Theta)\left(\frac{\mathrm{d}N}{\mathrm{d}\varphi}\right)_{\text{Coulomb or inelastic}}. \tag{7}$$

([14]) S. OLBERT: *Phys. Rev.*, **87**, 319 (1952); M. ANNIS, H. S. BRIDGE and S. O. OLBERT: *Phys. Rev.*, **89**, 1216 (1953).

Substituting from eq. (4) and (6),

$$\frac{\mathrm{d}N}{\mathrm{d}\varphi} = \frac{1}{2\pi}\left[1 + \eta_\Theta \sigma_T S_\Theta \cos\varphi\right] . \tag{8}$$

Making use of eq. (3), we obtain finally

$$\frac{\mathrm{d}N}{\mathrm{d}\varphi} = \frac{1}{2\pi}\left[1 - \alpha\eta_\Theta S_\Theta \sin\left(\Theta_p^* - \Theta_p\right)\cos\varphi\right] . \tag{9}$$

The product,

$$w = \eta_\Theta(A, p_i, x)\, S_\Theta(A, p_i) \sin\left(\Theta_p^* - \Theta_p\right) , \tag{10}$$

$$0 \leqslant w \leqslant 1 \tag{11}$$

is, in general, different for each event, and can be computed completely independently of any knowledge of the azimuthal angle $\varphi$. It can be regarded as the statistical weight of the event.

### 3. – Maximum likelihood estimation of the sign of $\alpha$.

The distribution over azimuth angle to be expected for the $i$-th event is, from eq. (9) and (10)

$$\mathrm{d}N_i = \frac{1}{2\pi}\left[1 - \alpha w_i \cos\varphi_i\right]\mathrm{d}\varphi_i . \tag{12}$$

We wish to estimate $\alpha$ from a set of $(w_i, \varphi_i)$ from the experimental data, using the maximum likelihood method. The likelihood function is

$$L = \ln\left[\prod_{i=1}^{n}\frac{1}{2\pi}\left(1 - \alpha w_i \cos\varphi_i\right)\right] , \tag{13}$$

$$= -\, n \ln(2\pi) + \sum_{i=1}^{n} \ln\left(1 - \alpha w_i \cos\varphi_i\right) . \tag{14}$$

The best estimate of $\alpha$ is the value which maximizes $L$. However, with a small number of scatters, each with a small weight, $L$ will have a rather broad maximum, meaning that we cannot expect a good estimate of the magnitude of $\alpha$. We therefore restrict ourselves to an estimation of the sign of $\alpha$, assuming its magnitude to be known much more precisely from other experiments (eq. (2)). From eq. (2) the lower limit on $\alpha$, which will be crucial for us, may be taken as

$$|\alpha| \geqslant .60 \tag{15}$$

if we combine the two experimental results and take the lower limit one standard deviation below the most probable value. Since we will always have

$$|\alpha w_1 \cos\varphi_i| < 1\,, \tag{16}$$

we can express eq. (17) more conveniently by expanding the logarithmic terms

$$L = -n\ln(2\pi) + \sum\left[-\alpha w_i\cos\varphi_i\left(1+\frac{\alpha w_i\cos\varphi_i}{2}+\right.\right. \tag{17}$$
$$\left.\left.+\frac{(\alpha w_i\cos\varphi_i)^2}{3}+\frac{(\alpha w_i\cos\varphi_i)^3}{4}+\ldots\right)\right].$$

We must choose the sign for $\alpha$ which makes $L$ greater. Let $L(+)$ and $L(-)$ be the values of $L$ corresponding to the choice of positive and negative $\alpha$, respectively. Then, the sign to be chosen for $\alpha$ must be the same as the sign of

$$L(+)-L(-) = -2|\alpha|\sum_i\left[w_i\cos\varphi_i\left(1+\frac{|\alpha|^2}{3}(w_i\cos\varphi_i)^2+\right.\right. \tag{18}$$
$$\left.\left.+\frac{|\alpha|^4}{5}(w_i\cos\varphi_i)^4+\ldots\right)\right].$$

If we let

$$D \equiv \frac{L(+)-L(-)}{2|\alpha|} = -\sum_i\left[w_i\cos\varphi_i\left(1+\frac{|\alpha|^2}{3}(w_i\cos\varphi_i)^2+\right.\right. \tag{19}$$
$$\left.\left.+\frac{|\alpha|^4}{5}(w_i\cos\varphi_i)^4+\ldots\right)\right],$$

then the sign of $D$ is the best choice of sign for $\alpha$. In most cases actually encountered,

$$(\alpha w_i\cos\varphi_1)^2 \ll 1 \tag{20}$$

therefore $D$ is given to a good approximation by

$$D \simeq -\sum_i w_1\cos\varphi_1\,, \tag{21}$$

which is an expression independent of the assumed magnitude of $\alpha$. If each experimenter with observations of $\Lambda^0$-decay proton scattering would report his value of $D$ as given by eq. (21), all the $D$'s available at any given time could be added together, the sign of the result giving the maximum likelihood estimate of the sign of $\alpha$ based upon all the available data.

**4. – Probability of experiment yielding the correct sign of $\alpha$.**

Let us suppose for the purpose of this discussion that we know both the magnitude and the sign of $\alpha$. To simplify the argument, let us assume that $\alpha$ is positive. We have seen in the preceding section that the outcome of the experiment will be decided by the sign of the random variable $D$ which is computed from the experimental observations. Suppose the experiment is repeated many times. Whenever $D$ is positive, we infer the correct sign of $\alpha$. However, since $D$ is a random variable subject to statistical fluctuations, it will sometimes be negative, in which cases we infer the wrong sign of $\alpha$. Let us compute what fraction of these repeated experiments will yield a correct result, hence $D > 0$.

The expected or average value, $\bar{D}$, of the random variable, $D$, can be found by integrating $D$ as given by eq. (19) over the distributions of eq. (12). We find

$$\bar{D} = \alpha \sum_i \frac{w_i^2}{2}\left(1 + \frac{\alpha^2 w_i^2}{4} + \frac{\alpha^4 w_i^4}{8}\right). \tag{22}$$

The variance, $V$, of $D$ is easily calculated, making use of the fact that the $\varphi_i$'s are statistically independent; hence, the variance of the sum of terms in eq. (19) is equal to the sum of the variances of the individual terms.

$$V(D) \equiv \overline{(D - \bar{D})^2} = \sum_i \frac{w_i^2}{2}\left(1 + \frac{5\alpha^4 w_i^4}{72}\right). \tag{23}$$

The standard deviation, $\sigma$, of $D$ is given by

$$\sigma(D) \equiv \sqrt{V(D)} = \sqrt{\sum_i \frac{w_i^2}{2}\left(1 + \frac{5\alpha^4 w_i^4}{72}\right)}. \tag{24}$$

Again, in most applications, it will be a good approximation to neglect the terms with higher powers of $\alpha w_i$.

We define a new random variable, $t$, which is proportional to $D$, and which measures $D$ in units of its standard deviation, $\sigma(D)$:

$$t \equiv \frac{D}{\sigma(D)}. \tag{25}$$

The average or expected value of $t$, is, dropping higher terms,

$$\bar{t} = \alpha \sqrt{\tfrac{1}{2} \sum_i w_i^2}. \tag{26}$$

The variance and standard deviation of $t$ are both unity as a consequence of its definition (eq. (25)).

Since $t$ is proportional to $D$, the experiment will yield the correct sign for $\alpha$ whenever $t$ is positive. We can roughly estimate the probability of finding $t>0$ if we approximate the distribution of $t$ by a Gaussian distribution with mean $\bar{t}$ (eq. (26)) and unit standard deviation. The probability of $t$ being between $t$ and $t+\mathrm{d}t$ is

$$p(t)\,\mathrm{d}t = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{(t-\bar{t})^2}{2}\right] \mathrm{d}t\,. \tag{27}$$

The probability that $t>0$ is

$$Q = \int_0^\infty p(t)\,\mathrm{d}t = \frac{1}{\sqrt{2\pi}} \int_0^\infty \exp\left[-\frac{(t-\bar{t})^2}{2}\right] \mathrm{d}t = \frac{1}{\sqrt{2\pi}} \int_{-\bar{t}}^\infty \exp\left[-\frac{x^2}{2}\right] \mathrm{d}x\,. \tag{28}$$

Making the obvious generalization, if the restriction of $\alpha$ being positive is dropped, the probability, $Q$, that the experiment will yield the correct sign for $\alpha$, given only the $w_i$'s and $|\alpha|$, is

$$Q = \frac{1}{\sqrt{2\pi}} \int_{-|\alpha|\sqrt{\frac{1}{2}\sum_i w_i^2}}^\infty \exp\left[-\frac{x^2}{2}\right] \mathrm{d}x\,. \tag{29}$$

Thus, if it is possible to make an estimate of $\sum_i w_i^2$ for a proposed experiment, the probability of obtaining the true sign of $\alpha$ can be estimated from eq. (29). For the events found in the Princeton cloud chamber,

$$\sum_i w_i^2 = 1.01\,, \tag{30}$$

$$Q \quad = 0.67 \quad \text{if} \quad |\alpha| = .60\,. \tag{31}$$

In other words, had we calculated $\sum_i w_i^2$ before going ahead with the complete analysis, we would have concluded that we had only a 67% chance of finding the correct sign from our data. This is one way of stating the confidence in our result.

## 5. – Confidence in the experimental sign of $\alpha$.

The confidence as calculated by eq. (29) or (31) does not make any use of the magnitude of the value actually found for $D$ in eq. (21). We would be inclined to believe that the result becomes more certain as the magnitude of

$D$ becomes larger. It would seem very unlikely to have a statistical fluctuation where $D$ is not only wrong in sign, but of large magnitude as well. In order to find the basis for a more definite statement of this type, let us go back to the probability of observing the set of $\varphi_i$'s actually found. This is the product over $i$ of the individual distributions given in eq. (12).

$$\Phi(\varphi_1, \varphi_2, \ldots, \varphi_n)\,\mathrm{d}\varphi_1, \mathrm{d}\varphi_2, \ldots, \mathrm{d}\varphi_n = \left[\prod_{i=1}^{n} \frac{1}{2\pi}(1 - \alpha w_i \cos\varphi_i)\right] \mathrm{d}\varphi_1, \mathrm{d}\varphi_2, \ldots, \mathrm{d}\varphi_n \,. \tag{32}$$

We now ask for the probability of the identical set of observations, $\varphi_i'$ but which would give the opposite sign for $\alpha$ in eq. (21). This would be the case if

$$\cos\varphi_i' = -\cos\varphi_i \,. \tag{33}$$

Then, we have

$$\Phi(\varphi_1', \varphi_2', \ldots, \varphi_n')\mathrm{d}\varphi_1', \mathrm{d}\varphi_2', \ldots, \mathrm{d}\varphi_n' = \left[\prod_{i=1}^{n} \frac{1}{2\pi}(1 + \alpha w_i \cos\varphi_i)\right] \mathrm{d}\varphi_1, \mathrm{d}\varphi_2, \ldots \mathrm{d}\varphi_n, \tag{34}$$

Eq. (32) gives the probability of a certain set of observations, $\varphi_i$, which, let us say for definiteness, give the correct sign for $\alpha$ in eq. (21). Eq. (34) gives the probability of another set of observations, $\varphi_i'$, which to an observer who knows only the absolute magnitude of $\alpha$ represent the same physical situation as the set $\varphi_i$; however, the set $\varphi_i'$ yield the wrong sign for $\alpha$ in eq. (21). The relative probability, $R$, of observing the set $\varphi_i$, hence the correct sign of $\alpha$, is

$$R = \frac{\Phi(\varphi_1, \varphi_2, \ldots, \varphi_n)}{\Phi(\varphi_1, \varphi_2, \ldots, \varphi_n) + \Phi(\varphi_1', \varphi_2', \ldots, \varphi_n')} \,. \tag{35}$$

Substituting from eq. (32) and (34), we have

$$R = \frac{\prod_i (1/2\pi)(1 - \alpha w_i \cos\varphi_i)}{\prod_i (1/2\pi)(1 - \alpha w_i \cos\varphi_i) + \prod_i (1/2\pi)(1 + \alpha w_i \cos\varphi_i)} \,. \tag{36}$$

Making use of eq. (13), (14), (17), (19), and (21),

$$R = \frac{1}{1 + \exp\left[-2\,|\alpha|\,|D|\right]} \simeq \frac{1}{1 + \exp\left[-2\,|\alpha|\,|\sum_i w_i \cos\varphi_i|\right]} \,. \tag{37}$$

Thus, for a given set $(w_i, \varphi_i)$, $R$ may be regarded as the probability that the sign found for $\alpha$ will be the correct one. $R$ may be considered another measure

of the confidence in our results which is based upon all the information available to us. This measure of confidence is very similar to that considered by BOLDT *et al.* ([2,3]). However, we regard $|\alpha|$ as fixed. Since only limits (eq. (15)) are known concerning its magnitude, we take the lower limit for $|\alpha|$ in eq. (37), thereby obtaining a conservative estimate of $R$. These authors compute what might be called an average value of $R$ by integrating over the allowed region of $|\alpha|$, giving equal weights to equal intervals of $|\alpha|$. It is not clear that any meaning can be attached to such an average since $|\alpha|$ has a unique, though unknown, value somewhere within the allowed limits. Only for the true value of $|\alpha|$ does $R$ have a precise meaning. By employing the lower limits $|\alpha|_{\min}$ for $|\alpha|$, we can at least say that

$$R \geqslant \frac{1}{1 + \exp\left[-2\,|\alpha|_{\min} \sum_i w_i \overline{\cos \varphi_i}\right]}. \tag{38}$$

## 6. – Example of experimental results.

Of 189 identified $\Lambda^0$ hyperons produced by 1.5 GeV $\pi^-$ and 2.8 GeV protons in carbon, iron, and lead plates of the Princeton cloud chamber ([5]), 117 decay protons penetrated one or more plates. In all, there were 87 carbon traversals, 57 iron traversals, and 101 lead traversals, all plates being one-half inch thick. From the identification analysis, the momentum, $p$, and the angles, $\Theta_p$ and $\Theta_p^*$ were known. It was found that the $\Lambda^0$'s observed in this experiment were essentially unpolarized ([15]). Due to multiple Coulomb scattering measurement error, and track distortions, every traversal resulted in a finite angle of scattering. Since it was not practical to compute $w_i$ in detail for every traversal, certain criteria, primarily based upon the projected scattering angle, $\psi$, were set up to eliminate most events where $w_i$ would be very small. Only events which met the following criteria were retained for further analysis:

$$\psi \geqslant 2.5^\circ, \tag{39}$$

$$\psi \geqslant \sqrt{2}\,\psi_{\text{r. m. s.}}, \tag{40}$$

$$\psi \leqslant \left(\frac{35}{KR}\right)^\circ, \tag{41}$$

where $\psi_{\text{r. m. s.}}$ is the calculated root-mean-square projected angle for the particular traversal, $K$ is the wave number, and $R$ is the nuclear radius of the

([15]) T. BOWEN, J. HARDY, Jr., G. T. REYNOLDS, G. TAGLIAFERRI, A. E. WERBROUCK and W. H. MOORE: *Phys. Rev. Lett.*, **1**, 11 (1958).

plate material nuclei. Eq. (39) eliminates most spurious scatters due to measurement errors and distortions; eq. (40) eliminates most Coulomb scatters; and eq. (41) eliminates most inelastic scatters ([11]). There remained for further analysis 21 scattering events, 2 in carbon, 6 in iron, and 13 in lead. The weights were rather small for most of the events in lead. The results were

$$D = -\sum_i w_i \cos \varphi_i = -0.142 \,, \tag{42}$$

$$R = 0.54 \quad \text{if} \quad |\alpha| = .60 \,. \tag{43}$$

Our data indicate a negative sign for $\alpha$, but with our set of data, we would only expect a 54% chance of deducing the correct sign; hence we attach no statistical significance to the result. Since the sign conventions may be confusing and differ from one author to another, it may be mentioned that negative $\alpha$ for us means that in the $\Lambda^0$ c.m.s. the spin vector of the proton is in the same direction as its direction of motion, *i.e.*, negative $\alpha$ means positive

TABLE I.

| Event no. | Material | $\eta$ | $S$ | $\sin(\Theta_p^* - \Theta_p)$ | $w_i$ | $w_i^2$ | $w_i \cos \varphi_i$ |
|---|---|---|---|---|---|---|---|
| 1 | C | .25 | .25 | .64 | .040 | .001 6 | —.025 |
| 2 | C | .92 | .50 | .98 | .451 | .203 0 | —.142 |
| 3 | Fe | .71 | .55 | .89 | .347 | .120 5 | +.058 |
| 4 | Fe | .99 | .55 | .94 | .523 | .273 5 | +.453 |
| 5 | Fe | .24 | .10 | .69 | .017 | .000 3 | —.017 |
| 6 | Fe | .78 | .55 | .93 | .399 | .159 2 | —.156 |
| 7 | Fe | .54 | .25 | .84 | .114 | .012 9 | —.114 |
| 8 | Fe | .67 | .55 | .99 | .365 | .133 3 | +.309 |
| 9 | Pb | .11 | .64 | .57 | .040 | .001 6 | +.015 |
| 10 | Pb | .31 | .28 | .96 | .113 | .012 8 | —.098 |
| 11 | Pb | .26 | .45 | .97 | .114 | .012 9 | +.065 |
| 12 | Pb | .30 | .40 | .87 | .104 | .010 9 | —.034 |
| 13 | Pb | .26 | .24 | .96 | .060 | .003 6 | —.032 |
| 14 | Pb | .32 | .45 | .99 | .143 | .020 3 | —.137 |
| 15 | Pb | .31 | .43 | .86 | .115 | .013 1 | —.078 |
| 16 | Pb | .24 | .23 | .74 | .041 | .001 7 | —.010 |
| 17 | Pb | .20 | .25 | .55 | .028 | .000 8 | —.019 |
| 18 | Pb | .36 | .35 | .59 | .074 | .005 5 | —.057 |
| 19 | Pb | .28 | .43 | .99 | .119 | .014 2 | +.119 |
| 20 | Pb | .21 | .22 | .92 | .043 | .001 8 | +.004 |
| 21 | Pb | .31 | .28 | .92 | .080 | .006 4 | +.038 |

$\sum_i \eta_i = 8.57$ $\qquad$ $\sum_i w_i^2 = 1.01$

$\sum_i w_i \cos \varphi_i = +.142$

helicity of the proton. The result of BOLDT *et al.* ([2,3]) gives a negative helicity with better statistical certainty than our result. However, the events found in the 30 inch propane chamber at Berkeley favor positive helicity ([1]), with $D = -2.79$, hence a statistical probability of giving the correct sign or $R \geqslant .96$ if $|\alpha| \geqslant .60$. Table I lists the 21 scatters to give an idea of the magnitude involved, and to illustrate that some cases do have reasonably high weights, $w_i$.

## 7. – Conclusions.

It has been shown that while information from the nuclear scattering of $\Lambda^0$-decay protons can be expected to determine the sign of the decay symmetry parameters, $\alpha$, the Princeton data by themselves are insufficient to make any choice of sign. However, if all experimenters would report their values for $\sum_i w_i^2$ and $\sum_i w_i \cos\varphi_i$ (defined in eq. (9) and (10), and employed in eq. (21), (29) and (38)), combined results for the sign of $\alpha$ and the confidence level could be computed.

* * *

We wish to thank G. FEINBERG and E. BOLDT for helpful discussions. The cloud chamber work was carried out in collaboration with. J. HARDY, W. H. MOORE, GEO. T. REYNOLDS, and G. TAGLIAFERRI, to whom we are greatly indebted. L. BLAU and J. BELINFANTE gave valuable assistance with the calculations.

RIASSUNTO (*)

Presentiamo un metodo statistico mediante il quale le informazioni fornite da un piccolo numero di scattering nucleari dei protoni di decadimento del $\Lambda^0$ possono essere usate per determinare il segno del parametro di asimmetria dovuto alla non-conservazione della parità nei decadimenti del $\Lambda^0$. Come esempio, applichiamo questa analisi a 21 scattering di protoni da iperoni $\Lambda^0$ decaduti nella camera a nebbia a molte piastre di Princeton. Poichè l'asimmetria che si può attendere in ogni scattering nucleare osservato è piccola, i nostri dati non forniscono un risultato statisticamente significativo. Tuttavia, se ogni ricercatore seguisse una analoga procedura con i propri dati, sarebbe possibile combinare i risultati in modo da determinare il segno con un alto grado di certezza.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. XX, N. 2 16 Aprile 1961

# Studies on Extensive Air Showers.

## Part II. – Sea Level Observations on the Fluctuations in the Densities of N-Particles, in Showers of the Same Size.

B. K. Chatterjee, G. T. Murthy, S. Naranan, B. V. Sreekantan and M. V. Srinivasa Rao

*Tata Institute of Fundamental Research - Bombay*

(ricevuto il 9 Gennaio 1961)

**Summary.** — A study of the fluctuations in the densities of N-particles in extensive air showers has been carried out with a set-up consisting of five N-detectors located at the centre of an array of scintillators. Air showers were grouped according to their size and the distance of the core from the N-detectors. In each group of showers of given size and core distance the observed frequency distribution in the number of N-detectors activated, shows the existence of large fluctuations in the densities of N-particles.

---

### 1. – Introduction.

The electron component is the most abundant one in air showers, and accordingly the lateral distribution, as well as the total number of electrons in individual showers, can be estimated fairly accurately for showers of size greater than $\sim 10^4$ particles, with a few detectors each of area $\sim 1$ m$^2$, arranged in a suitable array. On the other hand, the nuclear-interacting particles, (N-particles), constitute less than 1 to 2 per cent of the electrons, and therefore elaborate experimental set-ups are necessary even to detect them; the problem of determining the actual number of N-particles which have traversed any given area is very much more difficult. Consequently, it has not been practicable upto now to measure the densities of N-particles in individual showers. The lateral distribution of N-particles and their total number in showers of given size have therefore been generally obtained by a statistical method. In this procedure showers are grouped according to their size, and according to the distance, $r$, of the core from the N-detectors.

The density of N-particles at a distance $r$ from the core is then obtained from the ratio of the number of showers (in a given group, of fixed size and core distance $r$) in which the N-detectors are activated, to the total number of showers in that group; this procedure assumes that the N-particle density is uniquely given, and the probability whether the N-detector is activated or not, is governed by the Poisson law. This method is justifiable if large intrinsic fluctuations (outside the normally acceptable statistical variations) do not exist in the number of N-particles, from shower to shower. It is obvious, however, that this method will lead to erroneous results if wide fluctuations are present with appreciable frequency. The average values of density, and consequently of lateral distribution, and the total number of N-particles so obtained may not then be physically meaningful quantities.

In the experiments of NICOL'SKY *et al.* [1], LEHANE *et al.* [2] and in the work reported from this laboratory [3]—hereafter referred to as I—such a statistical procedure was used to obtain the variation of the total number of N-particles with the size of the shower. It was pointed out in I that the *average* number thus obtained of N-particles in a shower of given size may not be very meaningful, since it is rather ill-defined and since it has been got after ignoring intrinsic fluctuations, and that possible selection biases may have been introduced in the detection and analysis of the air shower, as a consequence of these fluctuations. Large fluctuations in energy flow, in the lateral distribution of electrons, and in the number of $\mu$-mesons in air showers have been reported recently by the Russian [4] and Japanese [5] groups. It is therefore important to determine experimentally whether wide fluctuations are present in the number of N-particles, and also the extent and frequency of such fluctuations, if they exist.

An experiment was designed specifically for this purpose, and some preliminary results have been obtained in a period of operation of three months at Bombay (sea level). In this paper we report the results obtained in this experiment. These results clearly show that large fluctuations occur in the density of N-particles in showers of the same size, with cores striking at a

[1] S. I. NICOL'SKY: *Proc. of the Oxford Conference on Extensive Air Showers* (1956) p. 19.

[2] J. A. LEHANE, D. D. MILLAR and M. H. RATHGEBER: *Nature*, **182**, 1699 (1958).

[3] B. K. CHATTERJEE, G. T. MURTHY, S. NARANAN, B. V. SREEKANTAN and M. V. SRINIVASA RAO: *Nuovo Cimento*, **18**, 1148 (1960).

[4] S. N. VERNOV, G. B. KHRISTIANSEN, A. T. ABROSIMOV, N. N. GORYUNOV, V. A. DMITRIEV, G. V. KULIKOV, YU. A. NECHIN, S. P. SOKOLOV, V. I. SOLOVIEVA, K. I. SOLOVIEV, Z. S. STRUGALSKY and B. A. KHRENOV: *Proc. of the Moscow Cosmic Ray Conference*, vol. 2 (Moscow, 1960), p. 7.

[5] S. FUKUI, H. HASEGAWA, T. MATANO, I. MIUYA, M. ODA, K. SUGA, G. TANAHASHI and Y. TANAKA: *Proc. of the Moscow Cosmic Ray Conference*, vol. 2 (Moscow, 1960), p. 30.

given distance from the N-detectors. The possible sources of these fluctuations and the experiments needed to separate out the relative contributions of the various sources are pointed out.

## 2. – Experimental details.

Experimental details and the method employed for analyzing the shower data have been given in I. One important difference between other experimental set-ups used ([1,2]), for the study of N-particles, and the present one is that *five* N-detectors each of area 0.4 m² have been clustered together at the centre of the air shower array of five scintillators: the scintillators were each of area 1 m². It is this feature that has enabled us to study the fluctuations in the densities of N-particles, in the manner discussed in this paper.

The experiment was in operation for three months at Bombay (sea level) and during this period about 5 000 showers were recorded; in about 600 of these, associated pulses from at least one of the five N-detectors were observed. The shower sizes recorded ranged from $10^4$ to $2.5 \cdot 10^6$ particles and the core distances extended upto 25 m from the centre of the array.

The calculations to determine shower size and core position were carried out with a mechanical analog computer; a lateral distribution function of the Nishimura-Kamata type was taken for these computations with a value of 1.2 for « $s$ », the age parameter. The error in size was 20 to 30 percent and the error in core position 1 to 2 m upto 10 m from the centre, and 3 to 5 m for cores beyond 10 m, as judged on the analog computer itself, for actual cases picked out arbitrarily from the total sample. About 5% of the showers deviated considerably from the assumed distribution function and those were excluded from the analysis on fluctuations of N-particles.

## 3. – Fluctuations in the densities of N-particles.

Showers of the same size $N_e$, with cores striking at a given distance $r$ from the centre of the N-detectors were grouped together. If the density of N-particles, $\Delta$ has the same value for all showers which constitute a group, then

$$\Delta(N_e, r) = \frac{1}{s\varepsilon \boldsymbol{m}} \ln \frac{T}{T - Q}, \tag{1}$$

where $s$ = area of each N-detector, $\boldsymbol{m}$ = total number of N-detectors, $\varepsilon$ = the efficiency of the detector for detecting N-particles; $T$ = total number of showers in the group, and $Q$ = number of showers in which one or more N-detectors were activated. For our set-up, $s = 0.4$ m², $\boldsymbol{m} = 5$, $\varepsilon = 0.25$.

The number, $F(\boldsymbol{n})$, of showers in which $\boldsymbol{n}$ out of $\boldsymbol{m}$ N-detectors are activated is given by the Poisson distribution:

$$F(\boldsymbol{n}) = T \cdot P(\boldsymbol{n}, \Delta) \,, \tag{2}$$

$$P(\boldsymbol{n}, \Delta) = {}^{m}C_{n}(1 - \exp[-s\varepsilon\Delta])^{n} \exp[(-s\varepsilon\Delta)(\boldsymbol{m} - \boldsymbol{n})] \,. \tag{3}$$

If the density of N-particles has a unique value for all showers in each group then $\Delta$ can be calculated from (1), and the frequency distribution from (2), and compared with the experimental distribution. This comparison is shown in Table I, for various groups of showers. (The grouping of showers according

TABLE I.

| Shower size ($N_e$) | Core distance (in m) | $\Delta$ | $T$ | $F(n)$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | $n=0$ | $n=1$ | $n=2$ | $n=3$ | $n=4$ | $n=5$ |
| $6.4 \cdot 10^4 \div$ $\div 1.6 \cdot 10^5$ | 2.6 ÷ 6.3 | 0.48 (0.56) | 277 | 218<br>218 | 47<br>53.5 | 10<br>5.23 | 0<br>0.26 | 2<br>0.006 | 0<br>— |
| | 6.4 ÷ 16 | 0.146 (0.16) | 442 | 411<br>411 | 29<br>29.7 | 0<br>0.86 | 2<br>0.012 | 0<br>— | 0 |
| $1.7 \cdot 10^5 \div$ $\div 4.0 \cdot 10^5$ | 0 ÷ 2.5 | 0.54 (0.61) | 59 | 45<br>45 | 10<br>12.5 | 4<br>1.38 | 0<br>0.07 | 0<br>— | 0<br>— |
| | 2.6 ÷ 6.3 | 0.50 (0.65) | 298 | 231<br>231 | 47<br>60 | 15<br>6.2 | 3<br>0.3 | 2<br>0.001 | 0<br>— |
| | 6.4 ÷ 16 | 0.18 (0.20) | 379 | 347<br>347 | 28<br>30.9 | 4<br>1.1 | 0<br>0.02 | 0<br>— | 0<br>— |
| $4.0 \cdot 10^5 \div$ $\div 1.0 \cdot 10^6$ | 0 ÷ 2.5 | 1.17 (1.68) | 103 | 58<br>58 | 22<br>35.2 | 11<br>8.7 | 6<br>1.1 | 5<br>0.06 | 1<br>0.001 6 |
| | 2.6 ÷ 6.3 | 0.94 (1.15) | 144 | 90<br>90 | 36<br>44.4 | 10<br>8.7 | 5<br>0.85 | 3<br>0.04 | 0<br>0.01 |
| $1.1 \cdot 10^6 \div$ $\div 2.5 \cdot 10^6$ | 2.6 ÷ 6.3 | 2.76 (3.0) | 24 | 6<br>6 | 8<br>9.6 | 7<br>6.1 | 0<br>1.94 | 1<br>0.31 | 2<br>0.02 |
| | 6.3 ÷ 16 | 1.13 (1.42) | 58 | 33<br>33 | 16<br>19.7 | 6<br>4.7 | 1<br>0.57 | 0<br>0.03 | 2<br>0.001 |

In each group, row 1: the observed distribution of $n$; row 2: the calculated distribution assuming equation (2).

to size and core-distance, and the intervals involved can be seen clearly from the Table itself.) The observed number of showers in which 2 or more detectors are activated ($n \geqslant 2$), is consistently higher than that calculated on the assumption of a Poisson distribution; in most of the groups the observed frequency of events with $n = 3$, 4 and 5, exceeds the expected number by a large factor. The fact that the shower sizes and core-distances have been taken over finite intervals can account only for a small factor ($\sim 2$) between the observed and expected frequencies of showers with $n \geqslant 3$.

The question may be raised whether cases in which two or more detectors are activated could not be, in the main, the result of cross-interference between detectors, *i.e.* a main interaction taking place in only one detector and the secondaries emitted at large angles from this interaction giving rise to further interactions in the adjacent detectors. This possibility is small as may be seen from the experimental observation that the frequency of events in which two *adjacent* N-detectors were activated, was the same as that of non-adjacent ones (in particular the extreme ones). This shows clearly that the double and multiple events were due to the incidence of separate nuclear active particles on the various detectors. This is essentially because the height of the N-detectors is small compared to their lateral extent and cross-interference can arise only due to secondaries emitted at very large angles which are consequently of lower energies and the efficiency of the N-detectors is low for such small energies.

The discrepancy between the observed and expected frequencies as seen from the Table can be understood only if $\Delta$ is not a constant, but fluctuates from shower to shower. Let $G(\Delta)\,\mathrm{d}\Delta$ represent the probability that in a given shower of a given size at a given distance from the core, the density of N-particles has a value between $\Delta$ and $\Delta + \mathrm{d}\Delta$. The frequency distribution $F(n)$ is then given by

$$f(n) = T\int_0^\infty G(\Delta)P(n, \Delta)\,\mathrm{d}\Delta\ , \tag{4}$$

where $P(n, \Delta)$ is given by (3). This distribution reduces to (2), when $G(\Delta)$ is assumed to be a $\delta$-function, which would be the case if there were no fluctuations. In principle it is possible to determine $G(\Delta)$ if the distribution function $F(n)$ is known accurately. However, the data are too meagre, especially for $n \geqslant 3$, to justify an attempt at an evaluation of $G(\Delta)$.

The rather close agreement between the observed and the expected number of showers for $n = 1$ and 2, when the data are normalized for $n = 0$, may, in the first instance, suggest that in a majority of the showers, the value of $\Delta$ is a constant, and close to that calculated from (1), and that it is only in a small fraction of showers that appreciable fluctuations exist. To estimate the magnitude and frequency of fluctuations on such a picture, we have drawn

the curves of $P(\Delta)$ *vs.* $\Delta$ for different values of $n$ ( = 0, 1, 2, 3, 4, 5) in Fig. 1. From these curves, one can estimate that the observed distributions in most of the shower groups of Table I can be accounted for if at least 5 to 10% of the showers fluctuate in density by as much as 10 times the density calculated from eq. (1). However, this interpretation is not unique, and many other types of fluctuations can also reproduce the observed distribution.

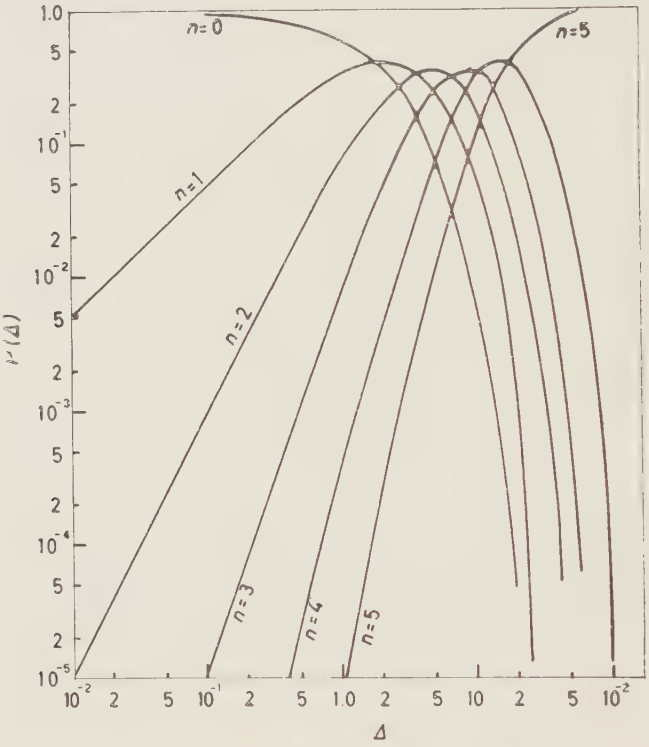


Fig. 1.

The estimation of an *average* density of N-particles, on the basis of eq. (1), presupposes that the distribution $F(n)$ is a Poissonian as represented by eq. (2) and (3). However, the experimental results are not consistent with such an assumption, since the observed number of events in which $n \geqslant 3$, is larger than that expected from eq. (1), (2) and (3).

It is therefore desirable to compute the average density without making any specific assumption regarding the nature of the distribution $F(n)$, *i.e.* according to the relation

$$\bar{\Delta} = \frac{\sum_{n=1}^{n=m} n F(n)}{s\varepsilon m T} \,. \tag{5}$$

The average densities calculated using (5) are given within brackets in the Table, below the values calculated from eq. (1). A comparison of the two sets of values shows that the densities calculated from (5) are consistently larger than those calculated from (1) by about 10 to 40%.

## 4. – Sources of fluctuations.

There are three different sources which can contribute to the fluctuations. They are:

1) fluctuations in the characteristics of nuclear interactions, particularly of the first collision; these are fluctuations in the inelasticity of collisions, in the multiplicity of mesons produced, in the energy transferred to $\pi^0$-mesons, etc.;

2) fluctuations in the level of the first collision, and in the inclination of the shower axis — in other words, fluctuations in the amount of atmosphere through which the shower has developed upto the level of observation; and

3) fluctuations due to the initiation of some of the extensive air showers by the heavy nuclei present in the primary cosmic radiation.

Nuclear emulsion data [6] on high-energy jets of energy $10^{12}$ to $10^{13}$ eV indicate that wide fluctuations exist in the characteristics of high energy interactions. No detailed quantitative data are available to enable an evaluation of the contribution of (1) to the fluctuations in the N-component.

As for (2), because of the fluctuations in the level of the first collision there is no one-to-one correspondence between the size of the shower and the energy of the primary particle. The distribution in the atmosphere of the levels of the first collision (which lead to air shower propagation), for a fixed size of shower observed, depends critically on the interaction mean free path, and on the energy spectrum of the primaries; and also on the longitudinal development and absorption of the electron component as a function of the primary energy. Until data on these parameters are available it is difficult to assess the magnitude of fluctuations in the N-component, due to fluctuations in the levels of shower origin. If the interaction mean-free-path in air is $\sim 70$ g cm$^{-2}$, calculations based on a simple model show that the fluctuations in the levels of shower-origin are not very important, but for larger values of the interaction mean free path [7] $(90 \div 100)$ g cm$^{-2}$ these fluctuations will result in large variations in the N-component.

---

(6) D. H. Perkins: *Progress in Elementary Particle and Cosmic Ray Physics*, vol. 5, Ch. IV (Amsterdam, 1960), p. 257.

(7) A. E. Brenner and R. W. Williams: *Phys. Rev.*, **106**, 1020 (1957).

The important role played by heavy primary nuclei in air shower phenomena has been stressed by PETERS ([8]). If one assumes that the primary charge spectrum observed at lower energies ($\sim 10^{13}$ eV) persist at higher energies as well, then one expects that at least 30% of the showers observed at sea level should be produced by $\alpha$-particles and heavier nuclei. The relative abundance of primaries of different mass number that can produce showers of the same size at sea level is given in Table II (COCCONI) ([9]).

TABLE II.

| Nature of the primary | H | He | CNO | Ne, Si | A, Fe |
|---|---|---|---|---|---|
| Mass number $A$ | 1 | 4 | 14 | 24 | 52 |
| Relative abundance (%) | 70 | 14 | 8 | 3 | 5 |

The relative abundances in the above Table have been obtained by COCCONI from the relative abundances of these nuclei at the top of the atmosphere after taking into account the decrease in the interaction mean free path with increase in mass number.

If it is assumed that a primary heavy nucleus of mass number $A$ and *total* energy $E_0$ is equivalent to $A$ nucleons each of energy $E_0/A$ as far as air shower development is concerned, and that the number $N_N$, of N-particles is proportional to $E^\alpha$ where $E$ is the energy of the nucleon generating the air shower, then it follows that $N_N \propto A(E_0/A)^\alpha$; or for a given energy $E_0$, $N_N$ is proportional to $A^{(1-\alpha)}$. If $\alpha = 1$, then $N_N$ is independent of $A$, *i.e.* there will be no fluctuations in the number of N-particles as a result of some of the showers being due to heavy nuclei. However, if $\alpha \neq 1$, then there will be a dependence of $N_N$ on the mass number of the primary particle. If $\alpha \sim 0.5$, then $N_N$ is very sensitive to $A$, and very large fluctuations in the number of N-particles may be expected, since the mass number of the primary can fluctuate by more than a factor of 50.

## 5. – Conclusions.

The present experiment has shown that it is not justifiable to assume that the number of N-particles is the same in all showers of the same size. There definitely exist wide fluctuations, though to determine the magnitude and frequency of such fluctuations more experimental observations are necessary.

([8]) B. PETERS: *Proc. of the Moscow Cosmic Ray Conference*, vol. **3** (Moscow, 1960).
([9]) G. COCCONI: *Extensive Air Showers, Handbuch der Physik*, vol. **45**.

Experimental results on the variation of the number of N-particles with shower size, etc. ([1-3]), which are based on the assumption that the total number of N-particles is always the same in showers of the same size except for normal statistical fluctuations must therefore be treated with caution.

In order to isolate out the various contributions which are due to causes listed previously and which give rise to fluctuations in the N-component, experiments are now being conducted with similar but more extensive air shower arrays at two different altitudes separated by an atmospheric depth of $\sim 200$ g cm$^{-2}$. In these experiments, in addition to data of the type reported in this paper, measurements are also being made on the age, the arrival direction *i.e.* the angle of the shower and the density of $\mu$-mesons. The number of N-detectors has also been doubled compared to the experiment reported herein. If fluctuations of appreciable magnitude in the N-component are seen in *showers of the same size, and of the same age*, then it would indicate that there exist sources of fluctuations other than fluctuations in the level of the first collision. To draw definite conclusions it is necessary to study fluctuations in showers of the same size, and the same age, over a wide range of sizes and of ages, which should be possible through experiments carried out at two atmospheric depths.

* * *

We have great pleasure in expressing our thanks to Professor M. G. K. Menon for his interest in this investigation and many helpful discussions. We are grateful to Professor S. Miyake for his valuable comments regarding the analysis of experimental data.

## RIASSUNTO (*)

Per mezzo di un'apparecchiatura costituita da cinque rivelatori di particelle N posti al centro di un gruppo di scintillatori si sono studiate le fluttuazioni della densità delle particelle N negli sciami estesi dell'aria. Questi sciami sono stati raggruppati secondo la loro grandezza e la distanza del « core » dai rivelatori di particelle N. In ogni gruppo di sciami di una data grandezza e di una data distanza del « core » la distribuzione di frequenza del numero di rivelatori di particelle N attivati, che si riscontra, mostra l'esistenza di ampie fluttuazioni delle densità delle particelle N.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. XX, N. 2 16 Aprile 1961

# Double Dispersion Relations for Deuteron Photo- and Electrodisintegration.

A. Martin and R. Vinh Mau

*CERN - Geneva*

(ricevuto il 16 Gennaio 1961)

**Summary.** — The deuteron photodisintegration matrix element, which exhibits, in the relativistic case, anomalous thresholds is studied in the nonrelativistic case, where the nucleons interact through a superposition of Yukawa or exponential potentials. For simplicity all spins are taken to be zero. A double dispersion relation is derived, in agreement with the latest results of Eden *et al.* The treatment is easily extended to electrodisintegration.

## 1. – Introduction.

Some interest has been recently devoted to the analytic properties of the deuteron photodisintegration matrix element, because the weakly bound character of the deuteron is expected to give rise to « anomalous thresholds » ([1-3]). Indeed, as was shown by de Alfaro and Rossetti ([4]), anomalous thresholds appear in a perturbation field theoretical treatment where the deuteron is regarded as an elementary particle; a Mandelstam representation may be written for the graphs containing at most one meson line with branch points given by the anomalous thresholds. De Alfaro and Rossetti have also shown that these features appear in a non-relativistic treatment of the deuteron photodisintegration at least for a given partial wave; in the same way one of

([1]) R. Karplus, C. M. Sommerfield and E. M. Wichmann: *Phys. Rev.*, **111**, 1187 (1958); **114**, 376 (1959).

([2]) R. Blankenbecler and Y. Nambu: *Nuovo Cimento*, **18**, 595 (1960).

([3]) L. Bertocchi, C. Ceolin and M. Tonin: *Nuovo Cimento*, **18**, 770 (1960).

([4]) V. de Alfaro and C. Rossetti: *Nuovo Cimento*, **18**, 783 (1960)

us (A.M.) has shown that for fixed real energy the analytic properties with respect to the angle are those predicted by the above mentioned Mandelstam representation (5).

We wish to derive in the present paper double dispersion relations for the matrix element under consideration in the non-relativistic framework. This is obtained by combining the analytic properties for fixed momentum transfer with the analytic properties for fixed, *almost real*, energy. It turns out that it is not difficult to extend, these results to off energy shell photodisintegration, *i.e.* electrodisintegration.

## 2. – The matrix element and its perturbation expansion with respect to the final state interaction.

As indicated in ref. (5) the non-relativistic matrix element for deuteron photodisintegration may be written in momentum space as

$$M(k, q, \cos\theta) = \int \psi_q^*(\boldsymbol{p})\, \psi_d\left(\boldsymbol{p} - \frac{\boldsymbol{k}}{2}\right) \mathrm{d}^3\boldsymbol{p} \,,$$

where $q$ is the relative momentum of the final nucleons, $k$ the momentum of the incident photon: $k$ and $q$ are related by the law of conservation of energy; $\psi_q$ is the ingoing final wave function, $\psi_d$ the deuteron wave function. The spins of all particles including the photon have been taken to be zero for simplicity.

For a neutron-proton potential of the type

$$V(r) = \int_\mu^\infty C(\alpha) \exp\left[-\alpha r\right] \mathrm{d}\alpha,$$

with

$$C(\alpha) < M\alpha^{1-\varepsilon} \,,$$

it has been shown by one of the authors (A.M.) (6) (and independently by Blankenbecler and Nambu (2)) that the deuteron wave function may be written

$$\psi_d\left(\boldsymbol{p} - \frac{\boldsymbol{k}}{2}\right) = \int_0^\infty \frac{\varrho(\sigma)\, \mathrm{d}\sigma}{(\boldsymbol{p} - \boldsymbol{k}/2)^2 + \sigma^2} \,,$$

(5) A. Martin: *Nuovo Cimento*, **19**, 344 (1961).
(6) A. Martin: *Nuovo Cimento*, **14**, 403 (1959).

with

$$\varrho(\sigma) = \frac{1}{N}\,\delta(\sigma - \alpha) + f(\sigma)\theta[\sigma - \alpha - \mu]\,,$$

$\alpha^2 = MB$ ($M$ nucleon mass, $B$ binding energy of the deuteron) and $f(\sigma)$ is const/$\sigma^{1+\varepsilon}$ for $\sigma \to \infty$.

A typical term of the integral representation of $\psi_d$ will be $1/((\boldsymbol{p} - \boldsymbol{k}/2)^2 + \sigma^2)$. So we shall first investigate the analytic properties of the quantity

$$M(k, q, \cos\theta, \sigma) = \int \frac{\mathrm{d}^3 p}{(\boldsymbol{p} - \boldsymbol{k}/2)^2 + \sigma^2}\,\psi_{\boldsymbol{q}}^*(\boldsymbol{p})\,.$$

Then, as $\varrho(\sigma)$ decreases rapidly enough at infinity we shall be able to obtain, by superposition, the analytic properties of $M(k, q, \cos\theta)$.

We shall now expand the final wave function in a perturbation series with respect to the final state interaction. Then

$$M(k, q, \cos\theta, \sigma) = \frac{1}{(\boldsymbol{q} - \boldsymbol{k}/2)^2 + \sigma^2} + \sum_1^\infty M_n(k, q, \cos\theta, \sigma)\,,$$

with, restricting ourselves for simplicity to the case of a pure Yukawa potential (*),

$$M_n(k, q, \cos\theta, \sigma) = \int \frac{1}{(\boldsymbol{p}_1 - \boldsymbol{k}/2)^2 + \sigma^2}\,\frac{1}{p_1^2 - q^2 - i\varepsilon}\cdot$$
$$\cdot\,\frac{1}{(\boldsymbol{p}_1 - \boldsymbol{p}_2)^2 + \mu^2}\,\frac{1}{p_2^2 - q^2 - i\varepsilon}\cdots\frac{1}{(\boldsymbol{p}_n - \boldsymbol{q})^2 + \mu^2}\,\mathrm{d}^3 p_1 \ldots \mathrm{d}^3 p_n\,.$$

Except for the fact that the first $\mu$ is replaced by $\sigma$, $M_n$ looks very much like an off energy shell $n$-th order scattering matrix element, with initial momentum $\boldsymbol{q}$ and final momentum $\boldsymbol{k}/2$. It is convenient to introduce the quantity $\Delta^2 = (\boldsymbol{q} - \boldsymbol{k}/2)^2$ which will be taken to be, by definition, the square of the momentum transfer. Actually it may be shown that except for additive and multiplicative constants the square of the four dimensional momentum transfer for the process $\gamma + \mathrm{d} \to \mathrm{N} + \mathrm{p}$ reduces to $\Delta^2$ in the non-relativistic limit (4). Taking into account the existence of a definite relationship between $q$ and $k$ we may write

$$M_n(q, k, \cos\theta, \sigma) = F_n(q^2, \Delta^2, \sigma) = f_n(k, \Delta^2, \sigma)\,.$$

---

(*) The extension to a more general case is very easy.

### 3. – Parametric representation of the $n$-th order matrix element.

In this section we closely follow the calculation of BOWCOCK and WALECKA (7) who derive, by Feynman's technique, a parametric representation of the scattering matrix element; the following result can be established

$$F_n(q^2, \Delta^2, \sigma) = \left(\frac{-1}{2\sqrt{\pi}}\right)^n \Gamma\left(\frac{n}{2}+1\right) \int_0^1 \mathrm{d}x_1 \int_0^{x_1} \mathrm{d}x_2 \ldots \int_0^{x_{2n-1}} \mathrm{d}x_{2n} \frac{1}{(a_1 \ldots a_n)^{\frac{3}{2}} (D_n)^{n/2+1}} ,$$

where

$$D_n = \left(\alpha + \frac{\gamma}{2}\right) q^2 + \left(\beta + \frac{\gamma}{2}\frac{k^2}{4} - \frac{\gamma}{2}\Delta^2 + \mu^2(x_n - x_{2n}) + \sigma^2 x_{2n} .$$

All the quantities depend on the $2n$ parametric variables and are given explicitly in Appendix I. $(a_1 a_2 \ldots a_n)$ vanishes on the border of the integration domain but this singularity of the integrand may be shown to be harmless. To study $D_n$ we shall use the following properties of $D_n$ derived in Appendix I:

1)
$$\begin{cases} -(1-x_n) < \alpha + \dfrac{\gamma}{2} < -\dfrac{(1-x_n)^2}{2}; & 0 < \beta + \gamma/2 < x_{2n}; \\ -(1-x_n) < \alpha + \beta + \gamma < -\dfrac{(1-x_n)^2}{2}; & \gamma < 0; \end{cases}$$

2)
$$\frac{2}{|\gamma|}\left[\mu^2(x_n - x_{2n}) + \sigma^2 x_{2n}\right] \geqslant (n\mu + \sigma)^2 .$$

### 4. – Analytic properties for fixed momentum transfer $\Delta^2$ real $> -[n\mu + \sigma]^2$.

Here, we have to make a choice for the kinematical relation between $k$ and $q$. Since the non-relativistic approximation has been made, and since the dynamical recoil effects in the photon absorption have been neglected, there is some arbitrariness in this choice. We have considered three cases:

*a*) non-relativistic kinematics, neglecting deuteron recoil

$$q^2 = Mk - MB ;$$

*b*) non-relativistic kinematics with deuteron recoil

$$q^2 = \frac{k^2}{4}\cdot\frac{2M}{M_\mathrm{d}} + Mk - MB > \frac{k^2}{4} + Mk - MB ;$$

(7) J. BOWCOCK and D. WALECKA: *Nucl. Phys.*, **12**, 371 (1959).

*c*) relativistic kinematics

$$k^2 = \frac{[4(q^2 + M^2) - M_d^2]^2}{16[q^2 + M^2]} .$$

All these formulae approximately coincide in the low energy region but they begin to differ strongly from each other for $|q^2| \sim M^2$. Out of these we shall choose the one which permits the simplest derivation of the desired properties.

*Case a*):

$$D_n = \left(\alpha + \frac{\gamma}{2}\right) Mk + \left(\beta + \frac{\gamma}{2}\right)\frac{k^2}{4} - \frac{\gamma}{2} \Delta^2 + \mu^2(x_n - x_{2n}) + x_{2n}\sigma^2 - MB\left(\alpha + \frac{\gamma}{2}\right),$$

with

$$-\frac{\gamma}{2}\Delta^2 + \mu^2(x_n - x_{2n}) + x_{2n}\sigma^2 - MB\left(\alpha + \frac{\gamma}{2}\right) > 0 ,$$

for $\Delta^2$ real $> -(n\mu + \sigma)^2$.

The possible singularities are given by $D_n = 0$. Assume first that $D_n$ for fixed values of the parametric variables, has two real roots in $k$; then from the signs of the coefficients they are both positive, and further, since $D_n = (\alpha + \gamma/2)q^2 + (\beta + \gamma/2)k^2/4 +$ positive quantity, they both correspond to real positive values of $q^2$. Unfortunately, it is not possible to guarantee that the roots are real because the discriminant is not always positive. If the roots are complex, their common real part is given by

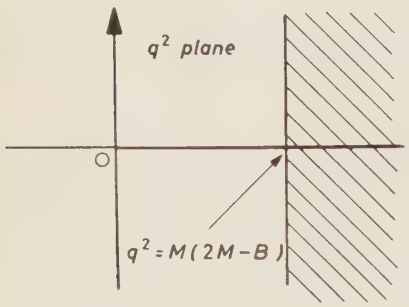


Fig. 2. – Minimum analyticity domain for case *a*. (Deuteron recoil neglected).

$$\mathrm{Re}\, k = -2M\frac{\alpha + \gamma/2}{\beta + \gamma/2} \geqslant 2M ,$$

so we get a minimum analyticity domain given by Fig. 1. May be one could enlarge this domain. However, the branch point $q^2 = 0$ may be shown to persist; it is an end-point singularity in the sense of EDEN [8].

*Case b*): Here a slight complication arises because we want to obtain, eventually, analytic properties with respect to the variable $q^2$: to a given value

[8] R. J. EDEN: *Phys. Rev.*, **119**, 1763 (1960).

of $q^2$ there correspond two values of $k$. However, $D_n$ is a well defined function of $k$:

$$D_n = \left[\left(\alpha + \frac{\gamma}{2}\right)\frac{2M}{M_d} + \left(\beta + \frac{\gamma}{2}\right)\right]\frac{k^2}{4} - Mk\left(\alpha + \frac{\gamma}{2}\right) + \text{positive quantity},$$

noting that $2M/M_d > 1$, and, hence that $(\alpha + \gamma/2)2M/M_d + (\beta + \gamma/2) < 0$, we easily see that $D_n$, as a function of $k$, has two real roots with opposite signs. Since, on the other hand, $D_n = (\alpha + \gamma/2)q^2 + (\beta + \gamma/2)k^2/4 + \text{positive quantity}$, they both correspond to positive values of $q^2$. We get two cuts in the $k$ plane:

$$k_1 \to +\infty, \qquad -\infty \to k_2,$$

with

$$k_1 = M_d\left[-1 + \sqrt{1 + \frac{2B}{M_d}}\right] > 0,$$

$$k_2 = M_d\left[-1 - \sqrt{1 + \frac{2B}{M_d}}\right] < -2M_d.$$

(see Fig. 2).

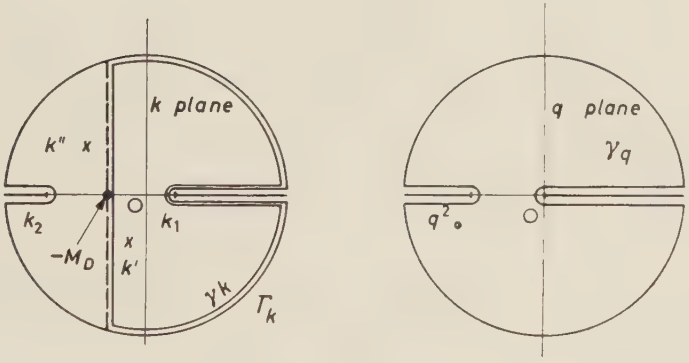


Fig. 2.

Let us now come back to the $q^2$ plane. We notice that from the relation

$$\frac{k^2}{4}\frac{2M}{M_d} + Mk - MB - q^2 = 0$$

we get that the sum $k' + k''$ of the two values of $k$ corresponding to a given $q^2$ is equal to $-2M_d$. One of the roots is to the right of the line $k = -M_d + i\lambda$ ($\lambda$ real), the other one to left. The line $k = -M_d + i\lambda$ correspond to

$$q^2 = -M\left[\frac{M_d}{2} + B\right] - \lambda^2\frac{M}{2M_d} = q_0^2 - \lambda^2\frac{M}{2M_d}.$$

Therefore, to make the correspondence $q^2 \leftrightarrow k$ a one to one correspondence we have to make a cut in the $q^2$ plane and select the value of $k$ connected with the physical region, *i.e.* the one with the largest real part. $F_n(q^2, \Delta^2, \sigma)$ for $\Delta^2$ real $> -(n\mu+\sigma)^2$ has therefore two cuts, one from 0 to $+\infty$ and the other one from $-\infty$ to $q_0^2 \simeq -M^2$ and no other singularity (Fig. 2).

*Case c*): Here $D_n$ can be expressed as a function of $q^2$

$$D_n = \left(\alpha + \frac{\gamma}{2}\right) q^2 + \left(\beta + \frac{\gamma}{2}\right) \frac{[4(q^2 + M^2) - M_d^2]^2}{64(q^2 + M^2)} + \text{positive quantity} .$$

One easily sees that $D_n$ vanishes for two values of $q^2$, one which is positive and the other one which is less than $-M^2$. So the analyticity domain in $q^2$ is exactly the same as in case *b*) (except for the replacement of $q_0^2 = -MM_d/2 - MB$ by $-M^2$).

If we now compare these three cases, we see that the *analytic properties for fixed momentum transfer in the region* $-M^2 < \mathrm{Re}\, q^2 < 2M^2$ *do not depend on the details of the kinematics*, which is rather encouraging. We shall now devote our attention to case *b*), because it is the one where it is easiest to get double dispersion relations. Indeed, we notice that in the $k$ plane $k^2$ and $q^2$ are both real positive along the two cuts. For $k^2$ and $q^2$ real positive we know the analytic properties in $\Delta^2$ from a preceding paper ([5]) and, as will be seen these properties may be extended to the case when $q^2$ and $k^2$ have a small imaginary part. On the other hand one may prove (Appendix I) that for $\Delta^2$ real $> -(n\mu+\sigma^2)+\varepsilon$, $D_n$ is larger than

$$\left|\frac{\mathrm{Im}\, k}{k}\right| \left[\frac{\varepsilon}{[n\mu + \sigma]^2} \sigma^2 x_n + \frac{(1 - x_n)^2}{2} |k|^2\right] ,$$

and from this one can deduce that we do not need any subtraction in the fixed momentum transfer dispersion relations.

## 5. – Analyticity in $\Delta^2$ for $q^2$ and $k^2$ nearly real, positive.

It has already been shown in ref. ([5]) that for $q^2$ and $k^2$ real and positive $M_n(k, q, \cos\theta, \sigma) = m_n(k^2, q^2, \Delta^2, \sigma)$ has the following analytic properties in $\Delta^2$: it has a cut from $-\infty$ to $-[n\mu+\sigma]^2$ and no other singularity. We want to extend in some way this result to the case where $k^2$ and $q^2$ have small imaginary parts (independent of each other). The following may be proved (Appendix II):

Provided the phases of $k^2$ and $q^2$ are such that

$$\left.\begin{aligned} \cos[\arg k^2] &\geqslant \frac{|k|^2 - 4\sigma^2}{|k|^2 + 4\sigma^2} + \varepsilon \\ \cos[\arg q^2] &\geqslant \frac{|q|^2 - \mu^2}{|q|^2 + \mu^2} + \varepsilon \end{aligned}\right\}, \qquad \varepsilon > 0,$$

One can show that $m_n$ is holomorphic in $\Delta^2$ outside a domain surrounding the cut which reduces to the cut as the arguments of $k^2$ and $q^2$ both go to zero (Fig. 3). This result could also be used to improve the rigor of the known proofs of Mandelstam representation for potential scattering.

## 6. – Double dispersion relations for non-relativistic kinematics with deuteron recoil.

Let us first consider the matrix element as a function of $k$ and $\Delta^2$. For $\Delta^2$ real $> -(n\mu+\sigma)^2$ we may write a dispersion relation in $k$, following the contour $\Gamma_k$. The behaviour at infinity is such that this contour integral reduces to the sum of the integrals around the two cuts. So

$$f_n(k, \Delta^2, \sigma) = \frac{1}{\pi} \oint_{\text{positive cut}} \frac{f_n(k', \Delta^2, \sigma)\,\mathrm{d}k'}{k' - k} + \frac{1}{\pi} \oint_{\text{negative cut}} \frac{f_n(k', \Delta^2, \sigma)\,\mathrm{d}k'}{k' - k},$$

if the path of integration is close enough to the cuts we may use the results of Section **5** and write

$$f_n(k, \Delta^2, \sigma) = \frac{1}{\pi^2} \oint_{\text{positive cut}} \mathrm{d}k' \oint_C \mathrm{d}\Delta'^2 \frac{f_n(k', \Delta'^2, \sigma)}{(k' - k)(\Delta'^2 - \Delta^2)} + \\ + \frac{1}{\pi^2} \oint_{\text{negative cut}} \mathrm{d}k' \oint_C \mathrm{d}\Delta'^2 \frac{f_n(k', \Delta'^2, \sigma)}{(k' - k)(\Delta'^2 - \Delta^2)},$$

where $C$ is indicated on Fig. 3. The paths of integration can be taken arbitrarily close to the cuts. We define in this way an analytic function of $k$ and $\Delta^2$ in the product of the cut planes. Let us now restrict ourselves to Re $k > -M_d$. Then a one to one correspondence between $k$ and $q^2$ may be established and $\mathrm{d}k/\mathrm{d}q^2$ is well defined and exists everywhere. Hence we deduce that $F_n(q^2, \Delta^2, \sigma)$ is holomorphic in the product of the $\Delta^2$ cut plane (cut: $-\infty \to -(n\mu+\sigma)^2$) and the $q^2$ cut plane (cuts: $-\infty \to +q_0^2 \simeq -M^2$,

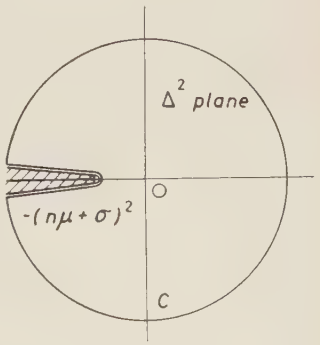


Fig. 3.

$0 \to +\infty$). Finally, assuming that no subtraction is needed as $|\Delta|^2 \to \infty$ (*), we may write

$$F_n(q^2, \Delta^2, \sigma) = \int_{-\infty}^{-(n\mu+\sigma)^2} \mathrm{d}\Delta'^2 \int_0^{\infty} \mathrm{d}q'^2 \frac{\varrho(q'^2, \Delta'^2)}{(\Delta'^2 - \Delta^2)(q'^2 - q^2)} + \\ + \int_{-\infty}^{-(n\mu+\sigma)^2} \mathrm{d}\Delta'^2 \int_{-\infty}^{-M[M_d/2+B]} \mathrm{d}q'^2 \frac{\varrho(q'^2, \Delta'^2)}{(\Delta'^2 - \Delta^2)(q'^2 - q^2)} .$$

So we get a Mandelstam representation containing an additional contribution from the left hand cut in $q^2$. In our opinion this additional contribution is *physically meaningless* because it corresponds to values of $|q|^2$ outside the non-relativistic region. The second term should be regarded, in fact, as a function of $\Delta^2$ *alone*, because its dependence on $q^2$ in the non-relativistic region is negligible. One might wonder, on the other hand whether it is possible to make this spurious cut disappear by multiplying $F_n(q^2, \Delta^2, \sigma)$ by some convenient factor, say $\sqrt{q^2 + M(M_d/2+B)}$; we were not able to do this but it might be so. The weight functions appearing in the representation have to be real because for real $\Delta^2$, $F_n(q^2, \Delta^2, \sigma)$ and $F_n(q^{2*}, \Delta^2, \sigma^2)$ are imaginary conjugate.

If we disregard the problem of convergence of the Born series (**), we may sum over $n$ and integrate over $\sigma$. We then get for the complete matrix element

$$F(q^2, \Delta^2) = \frac{1}{N(\alpha^2 + \Delta^2)} + \int_{-\infty}^{-(\mu+\alpha)^2} \mathrm{d}\Delta'^2 \int_0^{\infty} \mathrm{d}q'^2 \frac{\varrho(\Delta'^2, q'^2)}{(\Delta'^2 - \Delta^2)(q'^2 - q^2)} + \\ + \int_{-\infty}^{-(\mu+\alpha)^2} \mathrm{d}\Delta'^2 \int_{-\infty}^{-M(M_d/2+B)} \mathrm{d}q'^2 \frac{\varrho(\Delta'^2, q'^2)}{(\Delta'^2 - \Delta^2)(q'^2 - q^2)} .$$

If we compare this formula with the Mandelstam representation proposed by DE ALFARO and ROSSETTI [4] we see that apart from the last term (which in fact only depends on $\Delta^2$) and from the deuteron pole $1/(q^2+\alpha^2)$, they coincide. The absence of the pole in the $q^2$ variable is due to the expansion in

(*) This was shown to be the case for smooth superpositions of Yukawa potentials in ref. [5].

(**) Notice that here, as in the case of scattering [9], one can prove the convergence of the Born series for fixed real $\Delta^2 > -(\mu+\sigma)^2$, $|\mathrm{Im}\, k/k| > \varepsilon$, and $|k|$ large enough.

[9] A. KLEIN and C. ZEMACH: *Ann. Phys.*, **7**, 440 (1959).

power series of final wave function. One can make it appear by replacing the power series expansion by the Fredholm expansion of the final wave function as done for instance by JOST and PAIS ([10]). The Fredholm expression may be put in the form

$$\frac{\sum F'_n(q^2, \Delta^2)}{D(q^2)} .$$

The only singularities of $D(q^2)$ in the $q^2$ plane are the bound states of the $N$-$P$ potential plus a cut from 0 to $+\infty$. $F'_n(q^2, \Delta^2)$ is a sum of the $n$ first terms of the Born expansion with coefficients depending only on $q^2$ holomorphic, bound in the $q^2$ plane. The Fredholm numerator is absolutely convergent at least for physical values of $q^2$ and $\Delta^2((q-k/2)^2 < \Delta^2 < (q+k/2)^2)$. If we assume that the convergence property persists for non-physical values of $q^2$, $\Delta^2$, we can extract the deuteron pole $q^2 = -\alpha^2$ coming from the vanishing of $D(q^2)$ and write

$$F(q^2, \Delta^2) = \frac{1}{N(\alpha^2+\Delta^2)} + \frac{\lambda}{\alpha^2+q^2} + \int_{-\infty}^{-(\mu+\alpha)^2} \mathrm{d}\Delta'^2 \int_0^{\infty} \mathrm{d}q'^2 \, \frac{\varrho(\Delta'^2, q'^2)}{(q'^2-q^2)(\Delta'^2-\Delta^2)} +$$

$$+ \int_{-\infty}^{-(\mu+\nu)^2} \mathrm{d}\Delta'^2 \int_{-\infty}^{-M(M_\mathrm{d}/2+B)} \mathrm{d}q'^2 \, \frac{\varrho(\Delta'^2, q'^2)}{(q'^2-q^2)(\Delta'^2-\Delta^2)} .$$

It should be noted that the analytic properties of partial waves obtained from this formula are exactly those derived by DE ALFARO and ROSSETTI, in spite of the existence of the last term, because the cut $-\infty \to -M[M_\mathrm{d}/2+B]$ is « hidden » by the cut $-\infty \to -(\mu+\alpha)^2$. The fixed angle dispersion relations would only be modified for strongly forward angles such that $(1-\cos\theta) < (\mu+\alpha)^2/M^2$.

## 7. – Extension of the results to electrodisintegration.

Since we have already dealt with complicated kinematical relations we should expect that it is possible to still complicate slightly the problem by dealing with a virtual photon which appears for instance in the processes $\mathrm{e}+\mathrm{d} \to \mathrm{e}+\mathcal{N}+\mathrm{p}$ (Fig. 4) or $\mathcal{N}+\mathrm{p} \to \mathrm{d}+\mathrm{e}^+ + \mathrm{e}^-$. In the first case (electrodisintegration) the « photon mass » is imaginary; in the second case it is posi-

([10]) R. JOST and A. PAIS: *Phys. Rev.*, **82**, 840 (1951).

tive. It turns out that it is possible to write double dispersion relations for fixed negative mass of the photon, but not for fixed positive mass.

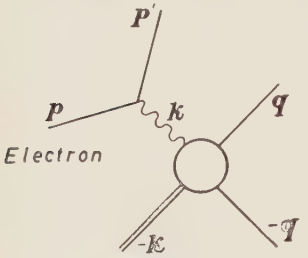

Fig. 4.

Let us consider the case of electrodisintegration. Since we are dealing with a non-relativistic $\gamma+\mathrm{d}\to\mathcal{N}+\mathrm{p}$ matrix element we have to use a non-covariant formalism where 3-momentum and mass are conserved in intermediate states. Hence $k=|\boldsymbol{p}-\boldsymbol{p}'|$ and the conservation of energy gives

$$\frac{q^2}{M}=\frac{k^2}{2M_\mathrm{d}}+E_p-E_{p'}-B=\frac{k^2}{2M_\mathrm{d}}+\delta-B\,,$$

the « photon mass square » which would appear in a covariant treatment is

$$|E_p-E_{p'}|^2-|p-p'|^2=\delta^2-k^2=-\lambda^2<0\,.$$

The physical situation corresponds to $\delta>0$. For fixed $\lambda^2$:

$$q^2=\frac{\delta^2}{4}\frac{2M}{M_\mathrm{d}}+\delta M+\lambda^2\frac{M}{2M_\mathrm{d}}-MB\,,$$

the matrix element can be considered as a function of $\delta$ and $\Delta^2$ (defined as in photodisintegration) or as a function of $q^2$ and $\Delta^2$ provided one makes a choice between the two possible corresponding values of $\delta$. Obviously, one should choose $\mathrm{Re}\,\delta>-M_\mathrm{d}$. One can proceed exactly as in photodisintegration. One first looks at the analytic properties in $\delta$ for fixed real $\Delta^2>-[n\mu+\sigma]^2$:

$$D_n=\left[\left(\alpha+\frac{\gamma}{2}\right)\frac{2M}{M_\mathrm{d}}+\left(\beta+\frac{\gamma}{2}\right)\right]\frac{\delta^2}{4}+M\delta\left|\alpha+\frac{\gamma}{2}\right|+$$
$$+\left[\left(\beta+\frac{\gamma}{2}\right)+\left(\alpha+\frac{\gamma}{2}\right)\frac{2M}{M_\mathrm{d}}\right]\frac{\lambda^2}{4}+\left|\frac{\gamma}{2}\right|\Delta^2+\mu^2[x_n-x_{2n}]+\sigma^2x_{2n}\,.$$

The discriminant of $D_n$ reads

$$M^2\left|\alpha+\frac{\gamma}{2}\right|^2-\left(\frac{M}{M_\mathrm{d}}\right)^2\lambda^2\left|\alpha+\frac{\gamma}{2}+\frac{M_\mathrm{d}}{2M}\left(\beta+\frac{\gamma}{2}\right)\right|^2+\text{positive quantity}\,.$$

For $\lambda^2<M_\mathrm{d}^2$ it is positive and $D_n$ vanishes for two real values of $\delta$. These two real values of $\delta$ correspond to *positive* values of $k^2(\lambda^2>0)$ and eventually to *positive* values of $q^2$. One of the roots in $\delta$ is less than $-M_\mathrm{d}$, the other one is larger than $-M_\mathrm{d}$. Therefore one can write double dispersion relations in $\delta$ and $\Delta^2$ and then in $q^2$ and $\Delta^2$. The cut in the $\Delta^2$ plane is the same as in photodisintegration. The cuts in the $q^2$ plane are $-\infty\to-(MM_\mathrm{d}/2)+$ $+\lambda^2(M/2M_\mathrm{d})-MB$ and $0\to+\infty$. These two cuts do not overlap for

$0 < \lambda^2 < M_d^2$. We should repeat here that the left hand cut in the $q^2$ plane has no real physical significance and that, on the other hand, one should not expect our treatment to be valid for values of $\lambda^2$ as large as $M^2$. In the case $\lambda^2 < 0$, which would happen in $\mathcal{N} + \mathrm{p} \to \mathrm{d} + \mathrm{e}^+ + \mathrm{e}^-$, the difficulty which cannot be overcome is that $k^2$ is not always positive for real $\delta^2$: one can show that for a parameter $x_n$ small enough one of the roots of $D_n$ is smaller than $|\lambda|$.

## 8. – Concluding remarks.

We have obtained double dispersion relations for photo- and electrodisintegration of the deuteron (the latter provided the 4-momentum transfer of the electrons is not exceedingly large). These double dispersion relations coincide with the Mandelstam representation if one disregards a spurious left hand cut starting at $q^2 \simeq -M^2$, far beyond the domain of validity of a non-relativistic approach. At the time this work is completed we understand that it had to be so, because in spite of the presence of anomalous thresholds the Mandelstam representation is valid for the fourth order terms obtained in a field-theoretical treatment of deuteron photodisintegration ([4]). According to R. J. Eden, Landshoff, Polkinghorne and Taylor ([11]) this ensures that it will be valid in all orders of perturbation theory.

Our result concerning electrodisintegration as well as the negative result concerning $\mathcal{N}$p radiative capture with emission of Dalitz pairs are in agreement with the general feeling one gets from field theory ([12]). This gives some support to the idea that whenever a Mandelstam representation is assumed for a photoprocess one should consistently extend it to the case where the photon is replaced by an electron producing a virtual photon.

* * *

The authors wish to thank Professors S. Fubini and V. Glaser for continued interest in this problem and Dr. Amati for some discussions.

## Appendix

The purpose of this appendix is to show the properties of $D_n$ listed in Section 3.

First, let us recall the definition of $\alpha$, $\beta$, $\gamma$, rewritten here in a slightly different form than in ([7]) and which is more convenient for the following cal-

---

([11]) R. J. Eden, P. V. Landshoff, J. C. Polkinghorne and J. C. Taylor: preprint.

([12]) V. Glaser: private communication.

culations,

$$\alpha = -\sum_{i=1}^{n} c_i + b_{n-1}\left(1 - \frac{b_{n+1}}{a_n}\right),$$

$$\beta = b_1 - \sum_{i=1}^{n} a_i \left(\frac{b_1 \dots b_i}{a_1 \dots a_i}\right)^2,$$

$$\gamma = -2b_{n+1} \prod_{i=1}^{n} \frac{b_i}{a_i},$$

with

$$\left.\begin{aligned} b_1 &= x_{2n}, \dots, b_{n+1} = x_n - x_{n+1}, \\ c_1 &= 1 - x_1, \dots, c_n = x_{n-1} - x_n, \end{aligned}\right\} \quad \text{satisfying the condition: } \sum_{i=1}^{n} c_i + \sum_{i=0}^{n} b_{i+1} = 1,$$

and

$$a_1 = c_1 + b_1 + b_2, \qquad a_i = c_i + b_{i+1} + b_i\left(1 - \frac{b_i}{a_{i-1}}\right).$$

**A.1. – Miscellaneous properties of the $a_i$'s and some of their combinations.**

In this Section, we shall prove some relations satisfied by the $a_i$'s which will be used in the next sections.

First, from their definition, it is very easy to see by recursion that

*a)* $$a_i \geqslant b_{i+1} + c_i \geqslant 0,$$

and *a fortiori*

*b)* $$1 - \frac{b_{i+1}}{a_i} \geqslant 0.$$

Now we want to show

*c)* $$1 - \frac{b_{n+1}}{a_n} - \prod_{i=1}^{n} \frac{b_i}{a_i} \geqslant 0.$$

Rewriting

$$1 - \frac{b_{n+1}}{a_n} - \prod_{i=1}^{n} \frac{b_i}{a_i} = \frac{1}{a_n}\left[a_n - b_{n+1} - \frac{b_1 \dots b_n}{a_1 \dots a_{n-1}}\right] =$$

$$= \frac{1}{a_n}\left[c_n + b_n\left(1 - \frac{b_n}{a_{n-1}} - \frac{b_1 \dots b_{n-1}}{a_1 \dots a_{n-1}}\right)\right],$$

(the latter equality is obtained simply by considering the definition of $a_n$).

Now, if we rearrange the bracket in the same manner, we may obtain immediately by recursion

$$1 - \frac{b_{n+1}}{a_n} - \prod_{i=1}^{n} \frac{b_i}{a_i} = \frac{c_n}{a_n} + \frac{b_n}{a_n}\frac{c_{n-1}}{a_{n-1}} + \frac{b_n}{a_n}\frac{b_{n-1}}{a_{n-1}}\frac{c_{n-2}}{a_{n-2}} + \dots + \frac{b_n}{a_n} \dots \frac{b_2}{a_2}\frac{c_1}{a_1}.$$

Since this is a series of positive terms, the above mentioned result follows.

**A.2. – Properties of $\alpha$, $\beta$, $\gamma$.**

*a)* $\quad \gamma \leqslant 0, \quad \beta + \frac{\gamma}{2} \leqslant x_{2n}$ .

The $a_i$'s and $b_i$'s being positive, these results are obvious from the definition of $\beta$ and $\gamma$.

*b)* $\quad \alpha + \frac{\gamma}{2} \geqslant -(1 - x_n)$ .

By definition

$$\alpha + \frac{\gamma}{2} = -\sum_{i=1}^{n} c_i + b_{n+1}\left(1 - \frac{b_{n+1}}{a_n} - \prod_{i=1}^{n}\frac{b_i}{a_i}\right).$$

But the last term is positive according to *c)* of Section A.1 and we have

$$\alpha + \frac{\gamma}{2} \geqslant -\sum_{i=1}^{n} c_i = -(1 - x_n) .$$

*c)* $\quad \beta + \frac{\gamma}{2} \geqslant 0, \quad \alpha+\beta+\gamma \leqslant -\frac{(1-x_n)^2}{2}, \qquad \alpha + \frac{\gamma}{2} \leqslant -\frac{(1-x_n)^2}{2}$ .

Let us introduce here $\alpha_n$, $\beta_n$, $\gamma_n$ defined by

$$\alpha(c_i, b_i) = \alpha_n(c_1, \ldots, c_n; b_1, \ldots, b_{n+1}),$$
$$\beta(c_i, b_i) = \beta_n(c_1, \ldots, c_n; b_1, \ldots, b_{n+1}),$$
$$\gamma(c_i, b_i) = \gamma_n(c_1, \ldots, c_n; b_1, \ldots, b_{n+1}),$$

where now $b_{n+1}$ may run from 0 to $\infty$, the range of the other $b_i$'s is kept to be (0, 1).

Now it is easy to verify, from the definition of $\alpha$, $\beta$, $\gamma$ and the results of Section A.1

i) $$\lim_{b_{n+1}\to\infty}\left(\beta_n + \frac{\gamma_n}{2}\right) = \beta_{n-1} + \frac{\gamma_{n-1}}{2},$$

i') $$\lim_{b_{n+1}\to\infty}(\alpha_n + \beta_n + \gamma_n) = \alpha_{n-1} + \beta_{n-1} + \gamma_{n-1},$$

ii) $$\frac{\partial}{\partial b_{n+1}}\left(\beta_n + \frac{\gamma_n}{2}\right) = \prod_{i=1}^{n}\frac{b_i}{a_i}\left[\prod_{i=1}^{n}\frac{b_i}{a_i} - \left(1 - \frac{b_{n+1}}{a_n}\right)\right],$$

which is always negative (see Section A.1 of this Appendix),

ii') $$\frac{\partial}{\partial b_{n+1}}(\alpha_n + \beta_n + \gamma_n) = \left(1 - \frac{b_{n+1}}{a_n} - \prod_{i=1}^{n}\frac{b_i}{a_i}\right)^2 \geqslant \frac{c_n^2}{a_n^2},$$

iii) $$\beta_1 + \frac{\gamma_1}{2} \geqslant 0 .$$

So we may conclude from i), ii), iii) that

$$\beta_n + \frac{\gamma_n}{2} \geqslant \beta_{n-1} + \frac{\gamma_{n-1}}{2} \ldots \geqslant \beta_1 + \frac{\gamma_1}{2} \geqslant 0 .$$

From i'), ii') we have

$$\alpha_n + \beta_n + \gamma_n = \alpha_{n-1} + \beta_{n-1} + \gamma_{n-1} - \int_{b_{n+1}}^{\infty} \frac{\partial}{\partial b_{n+1}} (\alpha_n + \beta_n + \gamma_n)\, \mathrm{d}b_{n+1} \leqslant$$

$$\leqslant \alpha_{n-1} + \beta_{n-1} + \gamma_{n-1} - \int_{b_{n+1}}^{\infty} \frac{c_n^2}{a_n^2}\, \mathrm{d}b_{n+1} \leqslant \alpha_{n-1} + \beta_{n-1} + \gamma_{n-1} - \frac{c_n^2}{a_n} ,$$

(because $a_n$ depends only linearly on $b_{n+1}$) and by recursion

$$\alpha_n + \beta_n + \gamma_n \leqslant - \sum_{i=1}^{n} \frac{c_i^2}{a_i} \leqslant - \sum_{i=1}^{n} \frac{c_i^2}{c_i + b_{i+1} + b_i} .$$

Now minimizing the last term we may get out:

$$\alpha_n + \beta_n + \gamma_n \leqslant \frac{- \left( \sum_{i=1}^{n} c_i \right)^2}{\sum_{i=1}^{n} (c_i + b_{i+1} + b_i)} = \frac{- (1 - x_n)^2}{1 + x_{n+1} - x_{2n}} \leqslant \frac{- (1 - x_n)^2}{2} ,$$

and also *a fortiori*

$$\alpha_n + \frac{\gamma_n}{2} \leqslant \frac{- (1 - x_n)^2}{2} .$$

*d*) $\alpha + \beta + \gamma \geqslant - (1 - x_n)$ .

This is obtained by combining *b*) and *c*).
Collecting then all the results we have

$$\gamma \leqslant 0 ,$$

$$- (1 - x_n) \leqslant \alpha + \frac{\gamma}{2} \leqslant - \frac{(1 - x_n)^2}{2} ,$$

$$0 \leqslant \beta + \frac{\gamma}{2} \leqslant x_{2n} ,$$

$$- (1 - x_n) \leqslant \alpha + \beta + \gamma \leqslant - \frac{(1 - x_n)^2}{2} .$$

A.3. – **Minimum of** $\Gamma = (2/|\gamma|)\,[\mu^2(x_n - x_{2n}) + \sigma^2 x_{2n}]$.

Rewriting $\Gamma$ in terms of the parameters $b_i$ we get

$$\Gamma = \frac{2}{|\gamma|}\left(\mu^2 \sum_{i=1}^{n} b_{i+1} + \sigma^2 b_1\right) \geqslant \sum_{i=0}^{n} \frac{1}{b_{i+1}}\left[\mu^2 \sum_{i=1}^{n} b_{i+1} + \sigma^2 b_1\right],$$

because $2/|\gamma| \geqslant \sum_{i=0}^{n} (1/b_{i+1})$ (see Appendix I of ref. (7)) or

$$\Gamma \geqslant \mu^2 \sum_{i=1}^{n} \frac{b_{i+1}}{b_1} + \sigma^2 + \left(\mu^2 \sum_{i=1}^{n} b_{i+1} + \sigma^2 b_1\right) \sum_{i=1}^{n} \frac{1}{b_{i+1}}.$$

Keeping first $\sum_{i=1}^{n} b_{i+1}$ and $b_1$ fixed, we may easily minimize this expression with respect to $b_2, \ldots, b_{n+1}$ this gives

$$\sum_{i=1}^{n} \frac{1}{b_{i+1}} \geqslant \frac{n^2}{\sum_i b_{i+1}},$$

and

$$\Gamma \geqslant \sigma^2 + \mu^2 n^2 + \mu^2 \sum_{i=1}^{n} \frac{b_{i+1}}{b_1} + n^2 \sigma^2 \frac{b_1}{\sum_{i=1}^{n} b_{i+1}}.$$

Minimizing now with respect to $\sum_{i=1}^{n} (b_{i+1}/b_1)$ we obtain

$$\Gamma \geqslant (n\mu + \sigma)^2,$$

so that

$$\frac{|\gamma|}{2} \Delta^2 + \mu^2(x_n - x_{2n}) + \sigma^2 x_{2n},$$

is always positive for $\Delta^2 > -(n\mu + \sigma)^2$.

A.4. – **Lower limit of** $D_n$ **and behaviour of** $f_n$ **as** $k^2 \to \infty$.

Let us first recall expression of $D_n$ as it is given in Section **4**

$$D_n = A\frac{k^2}{4} + M\left(\alpha + \frac{\gamma}{2}\right)k + C,$$

with

$$A = \left(\alpha + \frac{\gamma}{2}\right)\frac{2M}{M_d} + \beta + \frac{\gamma}{2} < \alpha + \beta + \gamma\,,$$

$$C = \frac{|\gamma|}{2}\Delta^2 + \mu^2(x_n - x_{2n}) + \sigma^2 x_{2n} - MB\left(\alpha + \frac{\gamma}{2}\right).$$

When $k$ is allowed to be complex we have

$$\operatorname{Re} D_n = -A\frac{|k|^2}{4} + \left|\frac{A}{4}\,2\operatorname{Re} k + M\left(\alpha + \frac{\gamma}{2}\right)\right| \operatorname{Re} k + C\,,$$

$$\operatorname{Im} D_n = \left[\frac{A}{4}\,2\operatorname{Re} k + M\left(\alpha + \frac{\gamma}{2}\right)\right] \operatorname{Im} k\,,$$

and

$$|D_n|^2 = |k|^2\left(Z + \frac{\operatorname{Re} k}{|k|^2}\,Y\right)^2 + Y^2\left|\frac{\operatorname{Im} k}{k}\right|^2,$$

with

$$Y = C - A\frac{|k|^2}{4}\,,$$

$$Z = \frac{A}{2}\operatorname{Re} k + M\left(\alpha + \frac{\gamma}{2}\right).$$

so that

$$|D_n| \geqslant \left|\frac{\operatorname{Im} k}{k}\right|\left(C - A\frac{|k|^2}{4}\right) > \left|\frac{\operatorname{Im} k}{k}\right|\left[C - (\alpha + \beta + \gamma)\frac{|k|^2}{4}\right].$$

Now we have seen that $-(\alpha + \beta + \gamma) > (1 - x_n)^2/2$ and it may be shown easily that

$$C > \frac{x_n \varepsilon \sigma^2}{(n\mu + \sigma)^2} \qquad \text{if} \qquad \Delta^2 > -(n\mu + \sigma)^2 + \varepsilon\,.$$

Finally

$$|D_n| > \left|\frac{\operatorname{Im} k}{k}\right|\left[\frac{\varepsilon\sigma^2}{(n\mu + \sigma)^2}\,x_n + \frac{(1 - x_n)^2}{8}|k|^2\right].$$

On the other hand one may notice that the the singularities occuring in the integrand due to the factor $(a_1 a_2 \ldots a_n)^{\frac{1}{2}}$ are harmless. Indeed one may easily see that

$$a_1 a_2 \ldots a_n > (b_1 + c_1)(b_2 + c_2) \ldots (b_n + c_n)\,,$$

and hence

$$\int_0^1 \mathrm{d}x_1 \int \ldots \int_0^{x_{2n-1}} \mathrm{d}x_{2n}\,\frac{1}{(a_1 a_2 \ldots a_n)^{\frac{1}{2}}} < \frac{[\Gamma(\frac{1}{2})]^n}{\Gamma(n + \frac{1}{2})}\,.$$

Combining these two results one can prove:

i) that the expansion in perturbation with respect to the final state interaction is bound by a geometric series;

ii) that a given term of the expansion, for real $\Delta^2 > 0$ vanishes as $|k|^2 \to \infty$, and that for $|k|^2$ large enough (with $\operatorname{Im} k/k > \varepsilon$) the expansion is convergent, a result similar to the one obtained in a different way by KLEIN [9] for the case of scattering.

## APPENDIX II

We wish to extend the analytic properties in $\Delta^2$, proved in ref. [5]), to the case where the initial and final energy are slightly complex. The procedure we used consisted in isolating the angular integrals. So for the $n$-th order term one has to compute

$$\int \mathrm{d}\Omega_1 \ldots \mathrm{d}\Omega_n \frac{1}{((k_f^2 + k_1^2 + \sigma^2)/2k_f k_1) - \cos\theta_{f1}} \cdot \frac{1}{((k_1^2 + k_2^2 + \mu^2)/2k_1 k_2) - \cos\theta_{12}} \cdots \frac{1}{((k_n^2 + k_i^2 + \mu^2)/2k_n k_i) - \cos\theta_{ni}},$$

when $k_f$ and $k_i$ are complex, it is convenient to integrate first on $\mathrm{d}\Omega_2, \ldots, \mathrm{d}\Omega_{n-1}$. Then the integral becomes

$$\int \mathrm{d}\Omega_1 \, \mathrm{d}\Omega_n \, \mathrm{d}x \frac{1}{((k_f^2 + k_1^2 + \sigma^2)/2k_f k_1) - \cos\theta_{f1}} \cdot \frac{1}{\left((k_1^2 + k_n^2 + (n-1)^2\lambda^2(x, k))/2k_1 k_n\right) - \cos\theta_{1n}} \frac{1}{((k_n^2 + k_i^2 + \mu^2)/2k_n k_i) - \cos\theta_{ni}},$$

with $\lambda(x, y) \geqslant \mu$; $x$ is written for $n-1$ parametric variables.

We now make use of the formula (see ref. [5])

$$\int \frac{\mathrm{d}\Omega_2}{(A - \cos\theta_{12})(B - \cos\theta_{23})} = 2\pi \int_1^\infty \frac{\mathrm{d}Z}{\sqrt{Z^2 - 1}} \frac{1}{[AB + Z\sqrt{A^2 - 1}\sqrt{B^2 - 1} - \cos\theta_{13}]},$$

and apply it to the first angular integration, and then to the second. We are led to an expression of the type

$$\frac{1}{(AB + Z\sqrt{A^2 - 1}\sqrt{B^2 - 1})\,C + Y\sqrt{C^2 - 1}\sqrt{(AB + Z\sqrt{A^2 - 1}\sqrt{B^2 - 1})^2 - 1} - \cos\theta}$$

We notice first that if $A$ is complex but such that

$$\operatorname{Re} A > 1, \quad \left| \frac{\arg\left(\sqrt{A^2-1}\right)}{\arg A} \right| < \frac{\operatorname{Re} A}{\operatorname{Re} A - 1/\operatorname{Re} A} .$$

Here

$$A = \frac{k_f^2 + k_1^2 + \sigma^2}{2k_f k_1} ,$$

and $\operatorname{Re} A > 1$ for any $k_1$ provided

$$\cos^2 \varphi_f > \frac{|k_f|^2}{|k_f|^2 + \sigma^2} ;$$

on the other hand $\arg A < |q_f|$. So by taking $q_f$ small enough one can make the argument of $AB + Z\sqrt{A^2-1}\sqrt{B^2-1}$ smaller than a fixed number, independent of $k_1$, $k_n$ and $Z$. Then one can show that the real part of $AB + Z \cdot \sqrt{A^2-1}\sqrt{B^2-1}$ can be made arbitrarily close to the quantity obtained by replacing $k_f$ by $|k_f|$, by choosing $\varphi_f$ small enough (irrespective of the value of $k_1$ and $k_n$). The latter quantity is known to be larger than

$$\frac{|k_f|^2 + k_n^2 + ((n-1)\mu + \sigma)^2}{2|k_f|k} > 1 .$$

One can then repeat the operation and prove than the integrand may be written as $1/(X - \cos\theta)$ with for

$$\cos^2 \varphi_f > \frac{|k_f|^2}{|k_f|^2 + \sigma^2} \qquad \cos^2 \varphi_i > \frac{|k_i|^2}{|k_i|^2 + \mu^2} ,$$

$$\arg(X) | < \lambda[|\varphi_i| + |\varphi_f|] ,$$

and

$$\operatorname{Re} X > \frac{|k_i|^2 + |k_f|^2 + (n\mu + \sigma)^2}{2|k_i||k_f|} - \varepsilon ,$$

with $\varepsilon \to 0$ as $q_i$ and $q_f$ go to zero. The important point is that $\lambda$ and $\varepsilon$ do not depend on the integration variables ($Z$, $Y$, $k_1$, $k_n$, etc.).

## RIASSUNTO (*)

L'elemento di matrice della fotodisintegrazione del deutone, che, nel caso relativistico, presenta soglie anomale, viene studiato nel caso non relativistico, in cui il nucleone interagisce tramite una sovrapposizione di potenziali di Yukawa o esponenziali. Per semplicità si suppone che tutti gli spin siano nulli. Si deriva una doppia relazione di dispersione, concordante con i più recenti risultati di EDEN *et al.* Il procedimento viene facilmente esteso all'elettrodisintegrazione.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. XX, N. 2 16 Aprile 1961

# On a Possible Enhancement of Relativistic Increase in Ionization.

P. Budini, L. Taffara and C. Viola

*Istituto di Fisica dell'Università - Trieste*
*Istituto Nazionale di Fisica Nucleare - Sottosezione di Trieste*

(ricevuto il 19 Gennaio 1961)

**Summary.** — A new method is described by which the relativistic increase of primary specific ionization in mixtures of elements could be augmented due to the reabsorption of Čerenkov radiation. The theory is given and applied to two examples.

## Introduction.

One of the most difficult problems, in experimental high energy physics is the distinguishing of the masses of relativistic particles and a promising approach to the solution of this problem, seems to be the combined measurement of momentum and ionization ([1,2]). The selectivity of the method obviously depends on the rapidity and amount of the relativistic increase of ionization after minimum. Recently it was shown, both theoretically ([3]) and experimentally ([1,2]), that primary ionization is preferable to total ionization as it presents higher relativistic increase. In this work, we will show that, there might be a further method to augment the relativistic increase.

([1]) R. C. Kepler, C. A. d'Andlau, W. B. Fretter and L. F. Hansen: *Nuovo Cimento*, **7**, 71 (1958).

([2]) C. Ballario, A. De Marco, R. D. Fortune and C. Verkerk: *Nuovo Cimento*, **19**, 1142 (1961). We wish to thank the authors for having communicated to us their results prior to publication.

([3]) P. Budini and L. Taffara: *Nuovo Cimento*, **4**, 23 (1956).

## 1. – The method.

Let us suppose to have a substance A with ionization frequency $\omega_A$ mixed with a substance B with ionization frequency $\omega_B$ such that

$$\omega_B > \omega_A . \tag{1}$$

When a sufficiently fast particle traverses the mixture, Čerenkov radiation will generally be emitted from B with frequency $\omega < \omega_B$. Because of (1), this radiation will ionize A and, if the density of A is sufficient, the ions will be formed in the neighbourhood of the trace, giving a new contribution to the primary ionization. If the density of A is too low or zero, the Čerenkov radiation emitted from B will go outside the medium or, at most, contribute to the excitation of B and to the formation of some ions of A far from the trace.

Turning to quantitative considerations, if $\varepsilon(\omega)$ represents the dielectric properties of the mixture and $\beta$ the velocity of the ionizing particle, the Čerenkov frequencies are those defined by the relation

$$\beta^2 \operatorname{Re} \varepsilon(\omega) > 1 .$$

In $\varepsilon(\omega)$, for not too high density, we can separate the component relative to the substance A from the component relative to B:

$$\varepsilon(\omega) = 1 + \delta_A(\omega) + \delta_B(\omega)$$

therefore in the limit $\beta \to 1$ the Čerenkov frequencies are those defined by

$$\operatorname{Re}[\delta_A(\omega) + \delta_B(\omega)] > 0 .$$

In the interval of frequencies from 0 to $\omega_A$ both $\delta_A$ and $\delta_B$ will be essentially positive (apart from small zones in vicinity of the lines) and we can then say that, in this interval, there will be emission of radiation both from A and from B. From $\omega_A$ to $\omega_B$, $\delta_A(\omega)$ will be essentially negative and $\delta_B(\omega)$ essentially positive. In the following, we suppose that, the density of A is lower than the density of B such that, the absolute value of $\delta_B$ prevails over that of $\delta_A$. We will then have in the interval $\omega_A \div \omega_B$ Čerenkov radiation emitted from B; since in this interval $\operatorname{Im} \delta_A > 0$ the Čerenkov radiation will be absorbed and generate primary ions of A. From $\omega_B$ to infinity, Čerenkov radiation will not be emitted.

To calculate this effect we start from the formula giving energy sent from the ionizing particle, in form of Čerenkov radiation at distances greater than $\varrho$

from the trace [4]:

$$\frac{\mathrm{d}W(\varrho)}{\mathrm{d}z} = \frac{e^2}{v^2} \int_{\text{Čerenkov}} \exp\left[-\frac{\omega}{c}\beta \operatorname{Im} \varepsilon(\omega)\varrho\right]\left[\beta^2 - \frac{\operatorname{Re}(\omega)}{|\varepsilon(\omega)|^2}\right]\omega \,\mathrm{d}\omega \,. \tag{2}$$

Then, the number of photons which have stopped inside $\varrho$, will be given by

$$\frac{\mathrm{d}N_{\check{\mathrm{C}}}(\varrho)}{\mathrm{d}z} = \frac{e^2}{\hbar v^2} \int_{\text{Čerenkov}} \left[1 - \exp\left[-\frac{\omega}{c}\beta \operatorname{Im} \varepsilon(\omega)\varrho\right]\right]\left[\beta^2 - \frac{\operatorname{Re} \varepsilon(\omega)}{[\varepsilon(\omega)]^2}\right] \mathrm{d}\omega \,. \tag{3}$$

In the case of the mixture just considered and if the lines of the element B are sufficiently narrow, we will have for $\omega_{\mathrm{A}} < \omega < \omega_{\mathrm{B}}$

$$\operatorname{Im} \varepsilon(\omega) = \operatorname{Im} \varepsilon_{\mathrm{A}}(\omega) + \operatorname{Im} \varepsilon_{\mathrm{B}}(\omega) \simeq \operatorname{Im} \varepsilon_{\mathrm{A}}(\omega)$$

and therefore

$$\frac{\mathrm{d}N_{\check{\mathrm{C}}}(\varrho)}{\mathrm{d}z} = \frac{e^2}{\hbar v^2} \int_{\omega_{\mathrm{A}}\ \text{Čerenkov}}^{\omega_{\mathrm{B}}} \left[1 - \exp\left[-\frac{\omega}{c}\beta \operatorname{Im} \varepsilon_{\mathrm{A}}(\omega)\varrho\right]\right]\left[\beta^2 - \frac{\operatorname{Re} \varepsilon(\omega)}{|\varepsilon(\omega)|^2}\right] \mathrm{d}\omega \,, \tag{4}$$

gives the number of ions of A formed by absorption of Čerenkov radiation inside a cylinder of radius $\varrho$ and axis the path of the primary particle.

## 2. – Discussion.

In order to discuss (4) we perform the integral with the mean value method:

$$\frac{\mathrm{d}N_{\check{\mathrm{C}}}(\varrho)}{\mathrm{d}z} = \frac{e^2}{\hbar v^2}\left[1 - \exp\left[-\frac{\overline{\omega}}{c}\beta \operatorname{Im} \varepsilon_{\mathrm{A}}(\overline{\omega})\varrho\right]\right] \int_{\omega_{\mathrm{A}}\ \text{Čerenkov}}^{\omega_{\mathrm{B}}} \left[\beta^2 - \frac{\operatorname{Re} \varepsilon(\omega)}{|\varepsilon(\omega)|^2}\right] \mathrm{d}\omega \,. \tag{5}$$

*a*) *Dependence on* $\varrho$. One can see from (5) that $\mathrm{d}N_{\check{\mathrm{C}}}(\varrho)/\mathrm{d}z$ as function of $\varrho$ increases linearly for small $\varrho$ and then goes exponentially to saturation If we call « radius of the Čerenkov trace » the distance $\varrho_{\check{\mathrm{C}}}$ inside which the part $(1 - e^{-1})$ of the total number of Čerenkov ions is formed, we have

$$\varrho_{\check{\mathrm{C}}} = \frac{c}{\overline{\omega}\beta}\left[\operatorname{Im} \varepsilon_{\mathrm{A}}(\overline{\omega})\right]^{-1} .$$

---

[4] P. Budini, L. Taffara and C. Viola: *Nuovo Cimento*, **18**, 864 (1960).

Then $\varrho_{\check{C}}$ is a linear function of the inverse of the density of the substance A. This could be one of the points subject to experimental verification.

The total number of Čerenkov ions

$$\frac{\mathrm{d}N_{\check{C}}}{\mathrm{d}z} = \frac{e^2}{\hbar v^2} \int_{\omega_A \text{ Čerenkov}}^{\omega_B} \left[\beta^2 - \frac{\operatorname{Re} \varepsilon(\omega)}{|\varepsilon(\omega)|^2}\right] \mathrm{d}\omega \tag{6}$$

is not dependent on the particular form of $\operatorname{Im} \varepsilon_A(\omega)$ (which represents the absorptive properties of A) and is only determined by the amplitude of the interval $\omega_A \mapsto \omega_B$, by $\beta$, and by $\varepsilon_{A+B}(\omega)$.

*b*) *Dependence on* $\beta$. If the density of A is sufficiently high such that $\varrho_{\check{C}}$ is less than the radius of the ionization trace, then (6) should be added to the number of primary ions for the mixture A + B, calculated with the method given in reference [3]. The variation of the relativistic increase due to the presence of (6) is the consequence of the dependence of (6) on $\beta$. For this purpose it is to be noted that the real interval of integration is not $\omega_A \mapsto \omega_B$, but the included intervals in which

$$\beta^2 \operatorname{Re} \varepsilon(\omega) - 1 > 0 .$$

Consequently $\beta^2 - (\operatorname{Re} \varepsilon(\omega)/|\varepsilon(\omega)|^2)$ is never negative. One sees therefore that $\mathrm{d}N_{\check{C}}/\mathrm{d}z$ is a positive monotone increasing function of $\beta$. Indeed, when $\beta$ increases, the integrand and the interval of integration also increase. Moreover for $\beta = 0$, $\mathrm{d}N_{\check{C}}/\mathrm{d}z$ is zero, and for $\beta = 1$ it reaches its higher value.

*c*) *Dependence on* $n_B$ (*number of* B *atoms per* cm$^3$). Let $\beta_0$ be a fixed value of $\beta$ near which primary ionization reaches the minimum. We take $\beta_0 = 0.937$ (which gives the minimum for low density matter). We suppose that the percentage of element A is so small that $\operatorname{Re}(\delta_A + \delta_B) \simeq \operatorname{Re} \delta_B$, and note that $\operatorname{Re} \delta_B$ is proportional to $n_B$ so that $\operatorname{Re} \delta_B = n_B F(\omega)$. Let us consider the two ratios, giving relativistic increase of pure primary ionization and of Čerenkov ionization:

$$\frac{\left(\frac{\mathrm{d}N_1}{\mathrm{d}t}\right)_{\beta=1}}{\left(\frac{\mathrm{d}N_1}{\mathrm{d}z}\right)_{\beta=\beta_0}} = \beta_0^2 \frac{\frac{1}{I_B} - \frac{1}{T} + H_1 \left(\log \frac{2mc^2}{I_B} - \frac{1}{2} \log n_B^2 \frac{C^2}{I_B^4} \bar{Q}_1^2 - 1\right) - K_1}{\frac{1}{I_B} - \frac{1}{T} + H_1 \left\{\log \frac{2mv^2}{I_B} - \frac{1}{2} \log \left[(1-\beta_0^2)^2 + \beta_0^4 n_B^2 \frac{C^2}{I_B^4} \bar{Q}_1^2\right] - \beta_0^2\right\} - K_1},$$

$$\frac{\left(\frac{\mathrm{d}N_{\check{C}}}{\mathrm{d}z}\right)_{\beta=1}}{\left(\frac{\mathrm{d}N_{\check{C}}}{\mathrm{d}z}\right)_{\beta=\beta_0}} = \beta_0^2 \frac{\int_{\text{Čerenkov}} \frac{n_B F(\omega)}{1 + n_B F(\omega)} \mathrm{d}\omega}{\int_{\text{Čerenkov}} \frac{\beta_0^2 n_B F(\omega) - (1-\beta_0^2)}{1 + n_B F(\omega)} \mathrm{d}\omega}.$$

The first one is calculated with the approximate formula reported in ref. (4), the second one with (6).

These two ratios depend on $n_B$ in very different way. Indeed the first one increase as $\log n_B$ or less, the second one nearly as $n_B$ (for $n_B F(\omega) \ll 1$, far from the line frequency); therefore, the Čerenkov ions could give a relativistic increase even if primary ions do not show increase for pure B because of the density effect. Such a case could also be realized in mixtures of liquids or gases and liquids.

Further, it is to be pointed out that the Čerenkov ions contribute essentially to primary ionization but very little or nothing to secondary ionization, since in most cases $\omega_B - \omega_A < \omega_A$ such that the photo-electrons of the Čerenkov ions do not have enough energy to ionize further. As a consequence of this, the effect is important especially for primary ionization and less for total ionization.

Summarizing, in order to put in evidence this effect, the mixture A + B should be composed of an element B, good Čerenkov emitter, with high ionization potential and an element A with low ionization potential and high absorption coefficient. Further, the density of A must be such that most of the Čerenkov photons, emitted from B, are absorbed inside the radius of the ionization trace.

### 3. – Applications.

As application of the above theory we have performed some calculations for mixtures of He+H and He+Alcohol.

*a*) The first mixture is formed by 90% of Helium and 10% of Hydrogen, at a pressure of 1 atmosphere and a temperature of 0 °C.

The number of total primary ions is then given by

$$\frac{dN}{dz} = \left(\frac{dN_1}{dz}\right)_{H+He} + \frac{dN_{\check{C}}}{dz} ,$$

where $(dN_1/dz)_{H+He}$ is the number of primary ions of H and He, calculated by formulae and methods described in (4).

In order to calculate the second term we have supposed that all lines of Helium in the interval $\omega_H - \omega_{He}$ may be represented by a single line and that in this very interval the contribution, of all other terms to dielectric constant may be considered a constant, that is

$$\text{Re}\ \varepsilon(\omega) = 1 + \frac{A}{\omega_1^2 - \omega^2} + a' . \tag{7}$$

In this case (6) can be exactly integrated and gives

$$(8)\qquad \frac{dN_{\check{C}}}{dz} = \frac{e^2}{\hbar v^2}\left[\left(\beta^2 - \frac{1}{1+a'}\right)(\omega_1 - \omega_\beta) + \right.$$

$$\left. + \frac{A}{2\sqrt{A/(1+a')+\omega_1^2}} \log \frac{\left(\sqrt{A/(1+a')+\omega_1^2}+\omega_1\right)\left(\sqrt{A/(1+a')+\omega_1^2}-\omega_\beta\right)}{\left(\sqrt{A/(1+a')+\omega_1^2}-\omega_1\right)\left(\sqrt{A/(1+a')+\omega_1^2}+\omega_\beta\right)}\right],$$

where:

$$A = \frac{4\pi e^2 Z n_{\mathrm{He}}}{m} f_1 ,$$

$Z$ = atomic number,

$f_1$ = oscillator strength of the first line of He,

$\omega_1$ = frequency of the first line,

$$a' = \frac{4\pi e^2 Z n_{\mathrm{He}}}{m}\left[\sum \frac{f_1}{\omega_1^2 - \omega_m^2} + \mathrm{Re}\int_{\omega_{\mathrm{He}}}^{\infty} \frac{f(\omega')\,d\omega'}{\omega'^2 - \omega_m^2 + ig\omega_m}\right] +$$

$$+ [\text{the same for } H] = 6.1\cdot 10^{-5},$$

$\omega_\beta$ is a frequency so that, in the interval $\omega_\beta - \omega_1$, $\beta^2\,\mathrm{Re}\,\varepsilon(\omega) - 1 \geqslant 0$; putting $\omega^* = \sqrt{\omega_1^2 - \beta^2 A/(1-\beta^2(1+a'))}$, it is $\omega_\beta = \omega^*$ if $\omega_{\mathrm{H}} \leqslant \omega^* \leqslant \omega_1$, $\omega_\beta = \omega_{\mathrm{H}}$ in the other cases.

Moreover, the absorption coefficient of H leads to $\varrho_{\check{C}} = 0.1$ cm. The two ratios

$$(9)\qquad \frac{N_1}{N_{1,0}} = \frac{(dN_1/dz)_{\mathrm{H+He}}}{[(dN_1/dz)_{\mathrm{H+He}}]_{\beta=\beta_0}},$$

and

$$(10)\qquad \frac{N}{N_0} = \frac{dN/dz}{(dN/dz)_{\beta=\beta_0}}.$$

are reported in Fig. 1.

It is seen that the reabsorption in H of Čerenkov radiation generated in He contributes considerably to the enhancement of primary ionization. At saturation ($\beta = 1$) it is $N/N_0 = 1{,}56$, as compared to $N_1/N_{1.0} = 1.28$.

*b*) As a second example we have considered the mixture of He and ethylalcohol. We have taken the data of the experiment of reference (²).

It is

Helium pressure = 775 mm Hg

Alcohol pressure = 39 mm Hg

Temperature = 20 °C.

The ionization potential of alcohol is 10.5 eV; for its absorption coefficient we took the data given by M. Ogawa and G. R. Cook (5). Those values lead

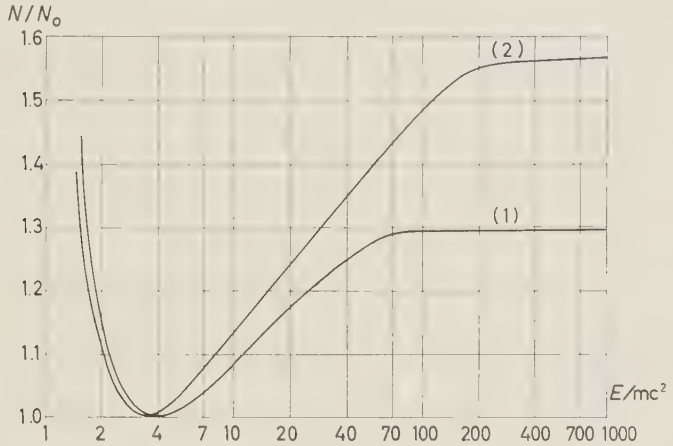


Fig. 1. – 1) Theoretical curve $N_1/N_{1,0}$. $N_{1,0}=3.78$. 2) Theoretical curve $N/N_0$. $N_0=4.03$.

to $\varrho_{\check{C}} = 10^{-3}$ cm and to a dielectric constant of alcohol which in the interval $\omega_{Al_2} \leftrightarrow \omega_1$ varies linearly. We have put therefore

$$\text{Re}\,\varepsilon(\omega) = 1 + \frac{A}{\omega_1^2 - \omega^2} + m\omega + a^* \,. \tag{11}$$

Also in this second example, with the further approximation $A/(\omega_1^2 - \omega^2) = A^*/(\omega_1 - \omega)$, (6) can be integrated,

$$\frac{\mathrm{d}N_{\check{C}}}{\mathrm{d}z} = \frac{e^2}{\hbar v^2}\left[\beta^2(\omega_1 - \omega_\beta) + \frac{\alpha}{1 + a^* + m\omega_1}\log\frac{\omega_1 - \omega_\beta + \alpha}{\alpha} + \right. \tag{12}$$

$$\left. + \frac{1}{m}\log\frac{1 + a^* + m\omega_\beta}{1 + a^* + m\omega_1}\right],$$

where

$$A^* = \frac{A}{\omega_1 + \omega_\beta},$$

$$\alpha = \frac{A^*}{1 + a^* + m\omega_1},$$

other symbols having the same meaning as in *a*).

(5) M. Ogawa and G. R. Cook: *Journ. Chem. Phys.*, **28**, 747 (1958).

To examine the contribution of the other lines of Helium and the influence of the finite breath of lines on the starting region of the Čerenkov radiation we have then considered an improved expression of dielectric constant:

$$\varepsilon(\omega) = 1 + \frac{8\pi e^2 n_{\mathrm{He}}}{m}\left[\frac{f_1(\omega_1^2-\omega^2)}{(\omega_1^2-\omega^2)^2+\eta_1^2\omega^2} + \frac{f_2(\omega_2^2-\omega^2)}{(\omega_2^2-\omega^2)^2+\eta_2^2\omega^2}\right] + a^* + m\omega\,,$$

where $n_{\mathrm{He}}$ = number of molecules of Helium per cm³,

$a^*$ is a medium value of all the other terms,

$$\eta = \frac{2e^2\omega^2}{3mc^3}\,,$$

and the other symbols have the usual meaning.

Practically the value of $\eta$ is much greater than the theoretical one, on account of which we have put, agreeing with COMPTON-ALLISON [6], $\eta_1 = \eta_2 = 10^{-3}\omega_{\mathrm{He}}$.

The two ratios $N_1/N_{1,0}$, $N/N_0$ are reported in Fig. 2, and compared with the experimental data. At the minimum, one finds $N_0 = 6.70$ experimentally, and $N_0 = 6.90$ theoretically.

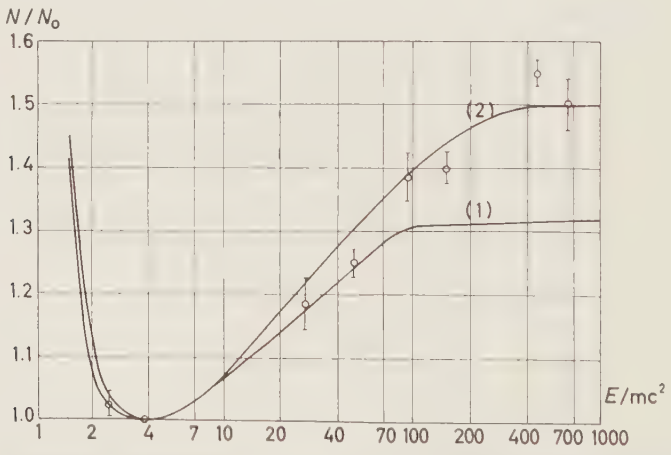


Fig. 2. – 1) Theoretical curve $N_1/N_{1,0}$. $N_{1,0} = 6.90$. 2) Theoretical curve $N/N_0$. $N_0 = 6.90$. ⊼ Experimental points (BALLARIO *et al.*). $N_0 = 6.70$.

With the simplified formula (12) one obtains a curve which differs from the one reported by less than 5%.

The effect of the finite breath of the lines is to shift the starting of Čerenkov effect towards higher energies.

(6) A. H. COMPTON and S. K. ALLISON: *X-Rays in Theory and Experiment* (1951), p. 302.

## 4. – Conclusion.

We have seen that the reabsorption of Čerenkov radiation in special mixtures of elements with different ionization potential should give an appreciable enhancement of relativistic increase of primary ionization. It would be interesting to prove experimentally the existence of such an effect. In fact it is to be hoped that if the effect exists, one could indicate, theoretically or experimentally, special mixtures or special dispositions of detectors that could amplify the effect, furnishing a favourable mean for particle detection in high energy physics.

* * *

This works has been sponsored by the United States Air Force European Office of Scientific Research.

RIASSUNTO

In misture di elementi in condizioni opportune si può avere un accrescimento dell'aumento relativistico della ionizzazione specifica primaria in seguito all'assorbimento da parte di uno degli elementi della radiazione di Čerenkov emessa dall'altro. Si dà la teoria del metodo e l'applicazione a due casi particolari.

IL NUOVO CIMENTO Vol. XX, N. 2 16 Aprile 1961

# Transverse Momentum of Particles Emitted in 4.2 GeV Proton-Proton Collisions (*).

M. H. Blue (**), J. J. Lord, J. G. Parks and C. H. Tsao

*Department of Physics, University of Washington - Seattle, Wash.*

(ricevuto il 20 Gennaio 1961)

**Summary.** — The transverse momentum spectra of pions and protons produced in inelastic 4.2 GeV proton-proton collisions are presented. Shapes of the observed momentum spectra seem to be independent of the number of particles produced in an interaction; however, the observed proton momenta tend toward higher values (by about a factor of 2) than the observed pion momenta. A transverse momentum spectrum for elastically scattered protons is compared to the proton spectrum from inelastic events. Good agreement is obtained for momenta above 300 MeV/c. Values of the proton transverse momentum less than about 100 MeV/c are much more probable for elastically scattered protons, while values around 150 MeV/c appear more probable for inelastically scattered protons.

## 1. – Introduction.

The cosmic ray observations of Nishimura [1] and a number of other groups [2-6] have shown that the transverse momentum of emitted shower

(*) Supported by the National Science Foundation, and by the joint program of the Atomic Energy Commission and the Office of Naval Research.

(**) Present Address, Lawrence Radiation Laboratory, Livermore, California, U.S.A.

(1) J. Nishimura: *Soryushiron Kenkyu*: **12**, 24 (1956) (in Japanese).

(2) S. F. Edwards, J. Losty, D. H. Perkins, K. Pinkau and J. Reynolds: *Phil. Mag.*, **3**, 237 (1958).

(3) O. Minakawa, Y. Nishimura, M. Tsuzuki, H. Yamanouchi, H. Aizu, H. Hasegawa, Y. Ishii, S. Takunaga, Y. Fujimoto, S. Hasegawa, K. Niu, K. Nishikawa, K. Imaeda and M. Kazuno: *Suppl. Nuovo Cimento*, **11**, 125 (1959).

(4) M. Schein, D. M. Haskin, E. Lohrman and M. Teucher: *Phys. Rev.*, **116**, 1238 (1959).

(5) R. R. Daniel, N. K. Rao, P. K. Malhotra and Y. Tsuzuki: *Nuovo Cimento*, **16**, 1 (1960).

(6) V. Y. Rajopadhye: *Phil. Mag.*, **5**, 537 (1960).

particles from high energy nuclear collisions is confined to a distribution of values with a distinct maximum between (300÷500) MeV/c. Some of these observations have been carried out at lower energies with accelerator particles ([5,6]) but most collisions studied involved collisions with emulsion nuclei rather than free protons. It was possible in the investigation reported herein to confine considerations largely to nucleon-nucleon collisions, to identify a significant fraction of the shower particles, and to measure their transverse momenta.

## 2. – Experimental procedure.

A stack of 600 μm thick Ilford G-5 pellicles was exposed to an internal beam of 4.2 GeV protons at the Berkeley Bevatron. The emulsions were processed and scanned for proton-proton collisions in the manner described by KALBACH *et al.* ([7]). In this experiment 113 collisions produced by 4.2 GeV protons were found which had the following characteristics which are required of free proton-proton collisions:

1) The nuclear event had an even number of tracks.

2) There was no recoil blob, decay electron, or other evidence of a collision with a bound nucleon.

3) An event can have no more than one slow proton track, and no proton tracks backward in the laboratory system.

Two prong events were classified as being inelastic only if they satisfied the additional criteria of either having an obvious momentum unbalance, or being an obvious case of pion production, or both. Once an event was selected as being an inelastic proton-proton like interaction, measurements were made of the scattering angle and, where possible, of the range, or blob density, and multiple scattering for all charged secondary tracks. The values of momentum times velocity, $pv$, determined from the multiple scattering or range data, were combined with values of the ionization loss determined from blob count data to identify the particles using the STERNHEIMER ([9]) relation between these quantities. Corresponding center of mass system values of the particle momentum, energy, and scattering angles were then computed.

As observed before ([7,8]) it is to be expected that about half of the selected collisions represent collisions with free protons and the rest peripheral collisions with larger emulsion nuclei. The selection procedure used in this experiment decreases the influence of internal nucleonic cascades by a large factor.

([7]) R. M. KALBACH, J. J. LORD and C. H. TSAO: *Phys. Rev.*, **113**, 330 (1959).
([8]) R. CESTER, T. F. HOANG and A. KERNAN: *Phys. Rev.*, **103**, 1443 (1956).
([9]) R. M. STERNHEIMER: *Phys. Rev.*, **91**, 256 (1953).

### 3. – Results.

The results of this experiment were first examined with respect to the transverse momentum for both pions and protons emitted in inelastic proton-proton collisions at 4.2 GeV. In Fig. 1 the transverse momentum distribution is given for all emitted charged pions. It is evident from the figure

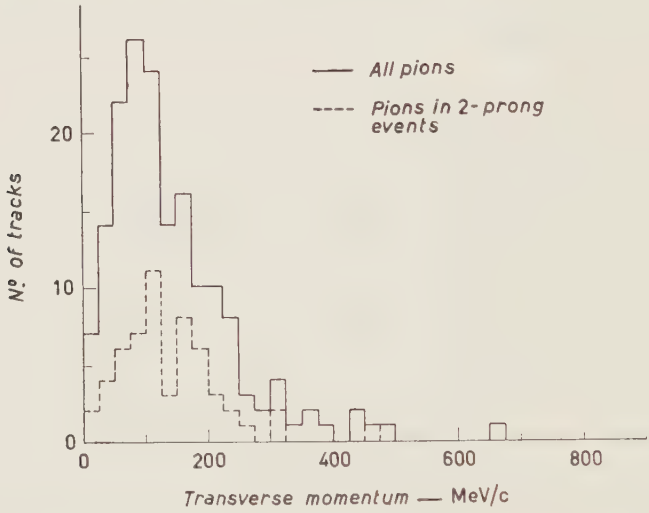


Fig. 1. – Distribution of transverse momentum of charged pions emitted in inelastic p-p collisions at 4.2 GeV. Solid line for all charged pions; dotted line for charged pions in events of two prongs.

that the most probable transverse momentum is about 90 MeV/c while the arithmetic mean was computed to be 142 MeV/c. In order to see if the multiplicity of pion production had any influence upon the transverse momentum, the distribution for 2 prong events is given by the dashed histogram in Fig. 1. The two histograms in Fig. 1 are sufficiently within statistical errors that there appears to be no significant dependence of the transverse momentum upon the multiplicity of pion production in this experiment.

The protons emitted in inelastic collisions were examined separately and a histogram of their transverse momentum distribution is given in Fig. 2. It is immediately evident that the transverse momentum distribution of the protons is similar in shape to that for the pions, Fig. 1, but shifted to higher momenta. The most probable transverse momentum of the protons is about 180 MeV/c while the arithmetic average is 265 MeV/c. Thus both the most

probable value and the mean value of the transverse momentum of the protons are about twice those for pions.

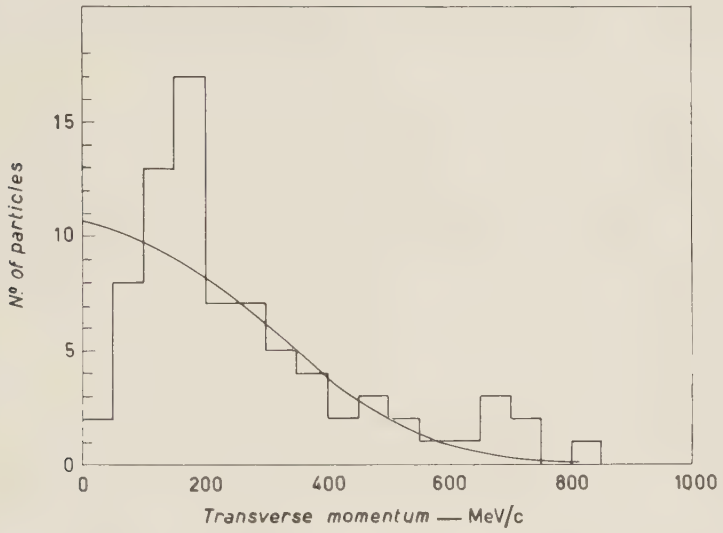


Fig. 2. – Distribution of transverse momentum of protons emitted in p-p collisions at 4.2 GeV. The histogram gives the distribution for protons from inelastic collisions. The smooth curve gives the distribution for protons from elastic collisions.

It proved interesting to compare the transverse momentum distribution for elastically scattered protons with that for inelastic ones. The transverse momentum distribution for elastically scattered protons is given by the solid curve in Fig. 2. This curve was obtained from the best fit optical model for the elastic scattering observations in this experiment. In this calculation a uniform sphere model was assumed with radius equal to 1.00 fermi and opacity equal to 0.90. As shown in Fig. 2, the transverse momentum distribution for the elastically scattered protons differs in a striking manner from that for the inelastic protons. The probability for low values of transverse momentum is much greater for elastically scattered protons than for inelastic ones. The mean value for the transverse momentum for elastically scattered protons was found to be 217 MeV/c which is to be compared with 265 MeV/c for inelastically scattered protons.

## 4. – Conclusions.

The general shape of the distribution of transverse momentum in Fig. 1 is rather similar to that found in high energy cosmic ray experiments. A typical distribution of transverse momentum of pions from cosmic ray exper-

iments, is given in Fig. 3. This distribution, taken from the work of MINIKAWA *et al.* [3], was obtained for neutral pions in nuclear reactions of energies in excess of 1000 GeV. It is striking that the distribution even at these high energies, Fig. 3, is very similar to that found in this experiment at 4.2 GeV, Fig. 1.

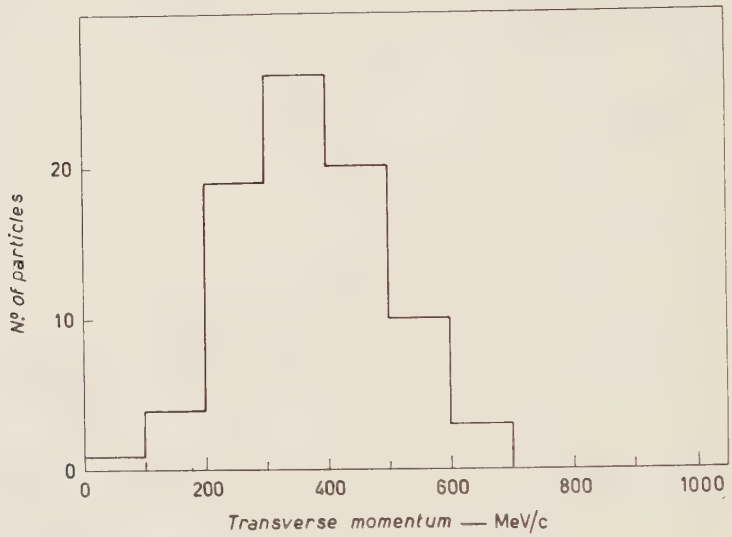


Fig. 3. – Distribution of transverse momentum of neutral pions from cosmic ray nuclear collisions in excess of 1000 GeV.

In this experiment the distribution of transverse momentum for pions, Fig. 1, was found to be measurably different from that for protons, Fig. 2. Both the most probable and average transverse momenta for protons were found to be about twice the corresponding values for pions.

A number of possible explanations of the low mean values for the transverse momentum distribution have been offered [10-12]. The hydrodynamical theory of meson production, the hypothesis of nuclear-Čerenkov radiation [10], the statistical theory of meson production [13], and considerations of single virtual pion exchange interactions all give qualitative explanations of transverse momentum distribution. There has been, however, no theoretical investigation of the difference between the distributions of transverse momenta for pions, Fig. 1, and protons, Fig. 2. In addition, the differences between the distri-

[10] G. YEKUTIELI: *Nuovo Cimento*, **13**, 446 (1959); **13**, 1316 (1959).
[11] V. S. BARASHENKOV and V. M. MALTSEV: *Nucl. Phys.*, **17**, 377 (1960).
[12] F. SALZMAN and G. SALZMAN: *Phys. Rev. Lett.*, **5**, 377 (1960).
[13] F. CERULUS and J. VON BEHR: *Nuovo Cimento*, **16**, 1046 (1960).

butions for elastically and inelastically scattered protons, Fig. 2, is quite striking. Here considerations of a single virtual pion exchange could give a plausible explanation of the differences between elastic and inelastic transverse momentum exchanges less than 150 MeV/c.

* * *

We would like to thank Drs. E. M. Henley, Y. B. Kim, S. H. Neddermeyer, G. E. Masek, and R. W. Williams for many helpful discussions. We are grateful to Drs. W. Chupp and E. J. Lofgren for emulsion exposures at the Bevatron. Dr. S. Kaneko was most helpful in discussions of the work of the Japanese emulsion group.

## RIASSUNTO (*)

Si presentano gli spettri dell'impulso trasversale dei pioni e protoni prodotti in collisioni anelastiche protone-protone di 4.2 GeV. Gli andamenti degli spettri dell'impulso osservati sembrano indipendenti dal numero di particelle prodotte in una interazione; tuttavia gli impulsi osservati per i protoni tendono a valori maggiori (per un fattore di circa 2) degli impulsi osservati per i pioni. Lo spettro dell'impulso trasversale per i protoni da scattering elastico viene confrontato con lo spettro per i protoni da eventi anelastici. Si ottiene un buon accordo per impulsi maggiori di 300 MeV/c. I valori dell'impulso trasversale inferiori a circa 100 MeV/c sono più probabili per protoni da scattering elastico, mentre valori attorno a 150 MeV/c appaiono più probabili per protoni da scattering anelastico.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. XX, N. 2 16 Aprile 1961

# Covariant Spin Operators and Associated Conservation Laws for a Spinor Field.

F. Calogero

*Istituto di Fisica dell'Università - Roma*
*Istituto Nazionale di Fisica Nucleare - Sezione di Roma*

(ricevuto il 25 Gennaio 1961)

**Summary.** — The covariant operators which may serve for the description of the spin of a Dirac field are considered, and their physical meaning discussed, with the aim of investigating the associated conservation laws in the case of an interacting spinor field. Particular attention is given to the case of a Dirac field interacting with an assigned electromagnetic field. Other operators, containing explicitly the coordinates, are then introduced, which also commute with the Dirac operator $D=\gamma_\mu \partial_\mu - m$ and lead therefore to conservation laws for a free spinor field. The validity of these conservation laws in the case of an interacting spinor field is again investigated, with a particular emphasis on the case when the source of interaction consists of an assigned electromagnetic field.

## 1. – Introduction.

Recently the problem of the covariant description of the spin of a Dirac field has received considerable attention (1). In this connection different covariant «spin operators» may be introduced, *i.e.* spin-differential operators constructed out of the $\gamma$-matrices and the differential operator $p_\mu = -i\partial_\mu$, which commute with the Dirac operator $D = i\gamma p - m$ and whose eigenvalues may therefore serve to label the eigenstates of $D$. The commutativity of these operators with $D$ also implies that the associated observables are constants of motion for a free field.

(1) V. Bargmann and E. Wigner: *Proc. Nat. Acad. Sc.*, **34**, 211 (1948); L. Michel and S. Wightman: *Phys. Rev.*, **98**, 1190 (1955); C. Bouchiat and L. Michel: *Nucl. Phys.*, **5**, 416 (1958); C. B. van Wyk: *Nuovo Cimento*, **10**, 854 (1958); H. P. Stapp: *Phys. Rev.*, **103**, 425 (1958); C. Frosdal and H. Überall: *Phys. Rev.*, **111**, 580 (1958).

In this paper the case of an interacting spinor field is instead considered, and the possibility is discussed, that the validity of some of the conservation laws mentioned above is still maintained in this case. The conditions for this to happen correspond to limitations to the source of interaction, and are discussed in detail.

In Section 2 the « spin operators » to be discussed in the following sections are introduced, and a number of algebraic relations are collected.

In Section 3 the physical meaning of the associated observables is discussed.

In Section 4 the case of an interacting spinor field is considered, and it is shown that the validity of some conservation laws is maintained, provided the source fields fulfil certain conditions.

In Section 5 the physically interesting case of a spinor field interacting with an electromagnetic field having an assigned space-time dependence is treated.

In Section 6 other operators are introduced, which contain explicitly the co-ordinates. Also these operators commute with the Dirac operator $D$, and lead therefore to conservation laws in the free field case. Their physical interpretation is discussed.

In Section 7 the case of an interacting spinor field is again considered, and the limitations on the source fields sufficient to maintain the validity of the conservation laws introduced in the preceding section are discussed. The case of a spinor field interacting with an assigned electromagnetic field is also treated in detail.

*Notation.* The metric $x \equiv (x_1, x_2, x_3, x_4 = ix_0)$ is used, $x^2 = x_\mu x_\mu = \boldsymbol{x}^2 - x_0^2$. Greek indices range from 1 to 4, latin indices from 1 to 3; summation over repeated indices is understood, except for dotted indices. Hermitian $\gamma$'s are used, $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$, $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$. A weak equality (symbol $\sim$) is said to hold between two spin-differential operators if they coincide when applied to a solution of the Dirac equation $D\psi = 0$. (Caution: weak equalities do not possess the multiplicative property, *i.e.* $A \sim B$ and $C \sim F$ do not imply $AC \sim BF$). A vector, tensor, etc., field will be termed hermitian when its hermiticity properties agree with the reality properties of an ordinary vector, tensor, etc. (for instance the vector field $A_\mu$ is called hermitian if $A_i^+ = A_i$, $A_4^+ = -A_4$).

## 2. – Spin operators: definitions and algebraic relations.

We call « spin operator » any spin-differential operator which commutes non trivially [2] with the Dirac operator $D = i\gamma p - m = \partial_\mu \gamma_\mu - m$. This name is justified by the fact that the eigenvalues of these operators may be used

[2] *I.e.*, is not constructed only out of $D$ and $p$.

to resolve the degeneracy of the eigenfunctions of the Dirac operator $D$, and we know that the internal degrees of freedom of a Dirac particle are connected with spin. Such operators, in particular fulfilling the requirement of covariance, have been recently investigated by many authors in the framework of a covariant description of spin (1). Our point of view is somewhat different, consisting in a search for spin-differential operators which not only commute with the Dirac operator $D$ and lead therefore to conservation laws for a free spinor field, but also lead to conservation laws in the more interesting case of an interacting spinor field. We will therefore limit the consideration to the simplest « spin operators », and in particular we will consider only « spin operators » which are linear in the differential operator $p_\mu$. The operators which are of interest for the covariant description of spin may instead require normalization factors, so that they may involve non-polynomial expressions in terms of $p_\mu$, *i.e.* they may represent non-local transformations.

The operators that we will consider in the following are:

$$\Sigma = -i\gamma_5\gamma_4\gamma_i p_i = (\boldsymbol{\sigma p}) , \tag{2.1}$$

$$M_\lambda = \frac{i}{2}\varepsilon_{\mu\nu\varrho\lambda}p_\mu\sigma_{\nu\varrho} = i\gamma_5\sigma_{\lambda\varrho}p_\varrho , \tag{2.2}$$

$$S_\lambda = -\gamma_5(p_\lambda + im\gamma_\lambda) , \tag{2.3}$$

$$W_{\alpha\beta} = -\gamma_5\gamma_\mu p_\nu\varepsilon_{\mu\nu\alpha\beta} = -W_{\beta\alpha} , \tag{2.4}$$

$$W'_{\alpha\beta} = \tfrac{1}{2}\varepsilon_{\alpha\beta\gamma\delta}W_{\gamma\delta} = -\gamma_5(\gamma_\alpha p_\beta - \gamma_\beta p_\alpha) = -W'_{\beta\alpha} . \tag{2.5}$$

The operator $\Sigma$ is, apart from a normalization factor, the usual non-covariant spin operator (3). The operator $M_\lambda$ is that introduced by Bouchiat and Michel (1), and the operator $W_{\alpha\beta}t_{\alpha\beta}$ is essentially that introduced by Michel and Wightman (1) with $s_\mu = P_\nu\varepsilon_{\nu\mu\alpha\beta}t_{\alpha\beta}$. We give below, for the convenience of the reader, a collection of algebraic relations fulfilled by these operators:

$$[D, M_\lambda] = 0 , \tag{2.6}$$

$$p_\lambda M_\lambda = 0 , \tag{2.7}$$

$$\gamma_\lambda M_\lambda = -3\gamma_5\gamma_\mu p_\mu \sim 3im\gamma_5 , \tag{2.8}$$

$$M_\lambda M_\lambda = -3p^2 \sim 3m^2 , \tag{2.9}$$

$$M_\lambda = S_\lambda - i\gamma_5\gamma_\lambda D \sim S_\lambda , \tag{2.10}$$

$$[D, S_\lambda] = 2\gamma_5 p_\lambda D \sim 0 , \tag{2.11}$$

(3) See, for instance, J. M. Jauch and F. Rohrlich: *The Theory of Photons and Electrons* (Cambridge, 1959), eq. (A2-50).

$$p_\mu S_\mu = -\gamma_5(p^2 + im\gamma_\mu p_\mu) \sim 0 , \tag{2.12}$$

$$\gamma_\mu S_\mu = -S_\mu \gamma_\mu = \gamma_5(\gamma_\mu p_\mu + 4im) \sim 3im\gamma_5 , \tag{2.13}$$

$$\tfrac{1}{2}[\gamma_\mu, S_\varrho] = \gamma_5(\gamma_\mu p_\varrho + im\,\delta_{\mu\varrho}) , \tag{2.14}$$

$$\tfrac{1}{2}\{\gamma_\mu, S_\varrho\} = m\gamma_5\sigma_{\mu\varrho} , \tag{2.15}$$

$$S_\mu S_\mu = p^2 + 4m^2 \sim 3m^2 , \tag{2.16}$$

$$S_\mu S_\varrho S_\mu \sim -m^2 S_\varrho , \tag{2.17}$$

$$\tfrac{1}{2}[S_\mu, S_\nu] = im(\gamma_\nu p_\mu - \gamma_\mu p_\nu) + im\sigma_{\mu\nu} , \tag{2.18}$$

$$\tfrac{1}{2}\{S_\mu, S_\nu\} = m^2\delta_{\mu\nu} + p_\mu p_\nu , \tag{2.19}$$

$$S_\beta \sim \tfrac{1}{2}\gamma_5\gamma_\alpha W_{\alpha\beta} , \tag{2.20}$$

$$mW_{\alpha\beta} = -iS_\mu p_\nu \varepsilon_{\mu\nu\alpha\beta} , \tag{2.21}$$

$$[D, W_{\alpha\beta}] = 0 , \tag{2.22}$$

$$p_\alpha W_{\alpha\beta} = 0 , \tag{2.23}$$

$$\gamma_\alpha W_{\alpha\beta} = -2i\sigma_{\nu\beta}p_\nu = 2\gamma_5 M_\beta \sim 2\gamma_5 S_\beta , \tag{2.24}$$

$$\gamma_\alpha\gamma_\beta W_{\alpha\beta} = \gamma_\alpha W_{\alpha\beta}\gamma_\beta = -6\gamma_\varrho p_\varrho \sim 6im , \tag{2.25}$$

$$W_{\alpha\beta}W_{\alpha\beta} \sim 6m^2 . \tag{2.26}$$

Note that $S_\lambda$ and $M_\lambda$ coincide weakly, eq. (2.10), and are therefore equivalent as long as one deals with free Dirac fields. Note also that all these operators commute with $D$ strongly, with the exception of $S_\lambda$ which only commutes weakly ([4]).

The fact that a spin differential operator $A$ commutes with $D$ will be sometimes referred to, in the following, by the statement that « $A$ is conserved for a free Dirac field ». This may be understood in the framework of the first quantized theory as meaning that the observable corresponding to $A$ is a constant of motion; or, in the framework of the second quantized theory, as meaning that the observable corresponding to the operator $\int \mathrm{d}\boldsymbol{x}\,\psi^+ A\psi$ is conserved.

Note also that each component of the « spin operators » introduced above is conserved in the free case; however the different components do not commute, and therefore only one (scalar, or pseudoscalar) eigenvalue may be assigned to label the eigenfunctions of $D$.

---

([4]) In general, any operator of the form $AD$ commutes weakly with $D$, since $[AD, D]=[A, D]D$. Therefore any operator which coincides weakly with an operator commuting with $D$ commutes weakly with $D$.

### 3. – Spin operators: physical interpretation.

We now want to investigate the physical meaning of the « spin operators » introduced in the preceding sections. This will serve for the physical interpretation of the conservation laws to be discussed in the following sections. In discussing the physical meaning of the conserved quantities we shall use a first quantized language. We emphasize however that this is a matter of language: the same discussion could be in fact rephrased in terms of the second quantized theory (5).

The method that we use for investigating the physical meaning of the « spin operators » is the following: we write them in terms of the usual operators of the non covariant formulation of the Dirac equation (6) and perform then a Foldy-Wouthuysen transformation (7), keeping only the terms which are « even », and putting $\beta = 1$ (*i.e.* we keep only the part of the operator which contributes to the mean value in a state of positive energy). The result may then be read out directly, since it is then written in terms of operators having a direct physical meaning, $\boldsymbol{\sigma}$ standing for the spin, $\boldsymbol{p}$ for the momentum and $\boldsymbol{x}$ for the position of the particle.

In the following we will, for simplicity of language, term an operator $A$ directly with the name obtained according to this « physical interpretation ». For instance, if the operator $\boldsymbol{A}$ is conserved and reduces to $\boldsymbol{\sigma}$ after the procedure above, we will simply say that « the spin is conserved ». We emphasize that this is only an abbreviation for the sentence « there exists an operator $\boldsymbol{A}$, which is a constant of motion and whose mean value in a free state containing one (positive energy) particle coincides with the mean value of the operator representing the « mean » (in the sense of reference (7)) spin ». Notice that this convention does not lead to a biunivocal correspondence between operators and names, since different operators may have the same physical interpretation (5). Note also that in general the physical interpretation of $AB$ is different from the product of the physical interpretations of $A$ and of $B$. This may happen also when $A$ and $B$ coincide, the typical example being the oper-

(5) In the second quantized theory the difficulties in the physical interpretation have their roots in the noncommutativity of the operators in question with the particle number. But a consistent procedure to derive a physical interpretation for any operator of the type we are considering may be given, and consists essentially in the elimination of the non commuting part, i. e. exactly the part which in the first quantized theory gives rise to zitterbewegung effects and is isolated by means of the Foldy-Wouthuysen transformation. See B. F. Touschek: *Suppl. Nuovo Cimento* (to be published).

(6) This first step is of course unessential.

(7) L. L. Foldy and S. A. Wouthuysen: *Phys. Rev.*, **78**, 29 (1950).

ator $\alpha_k$, whose physical interpretation is the $k$-th component $v_k$ of the velocity, and whose square is 1.

The results are collected in Table I, in which we have also essentially copied down some lines of Table I of reference (7), in order to fix the notation.

TABLE I. – *Physical meaning of « spin operators ».* 1) Covariant notation; 2) noncovariant notation; 3) part « even » and with positive energy ($\beta=1$) of the operator, after the Foldy-Wouthuysen transformation.

$$m^2 = E^2 - \boldsymbol{p}^2 = E^2(1-v^2); \quad \hat{\boldsymbol{p}} = \boldsymbol{p}/|\boldsymbol{p}|; \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}_p + \boldsymbol{\sigma}_T, \quad \boldsymbol{\sigma}_p = (\boldsymbol{\sigma}\hat{\boldsymbol{p}})\hat{\boldsymbol{p}}.$$

| 1) | 2) | 3) |
|---|---|---|
| $D\psi = (i\gamma p - m)\psi = 0$ | $(\boldsymbol{\alpha}p + \beta m - E)\psi = 0$ | $(E-E)\psi' = 0$ |
| $-\gamma_4$ | $\beta$ | $m/E$ |
| $\sigma_{\mu\nu} = \frac{1}{2i}[\gamma_\mu, \gamma_\nu]$ | $\boldsymbol{\sigma}(\sigma_i = \frac{1}{2}\varepsilon_{ijk}\sigma_{jk})$<br>$\boldsymbol{\alpha}(\alpha_i = \sigma_{i4})$ | $\boldsymbol{\sigma}_p + (m/E)\boldsymbol{\sigma}_T$<br>$v\hat{\boldsymbol{p}}$ |
| $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$ | $-(\boldsymbol{\sigma}\hat{\boldsymbol{p}})(\boldsymbol{\alpha}\hat{\boldsymbol{p}})$ | $-v(\boldsymbol{\sigma}\hat{\boldsymbol{p}}) = -v\lvert\boldsymbol{\sigma}_p\rvert$ |
| $\Sigma = -i\gamma_5\gamma_4\gamma_i p_i$ | $\Sigma = (\boldsymbol{\sigma}\boldsymbol{p})$ | $(\boldsymbol{\sigma}\boldsymbol{p}) = (\boldsymbol{\sigma}_p\boldsymbol{p})$ |
| $M_\lambda = \frac{i}{2}\varepsilon_{\mu\nu\varrho\lambda}p_\mu\sigma_{\nu\varrho} = i\gamma_5\sigma_{\lambda\varrho}p_\varrho$ | $\boldsymbol{M} = E\boldsymbol{\sigma} - i\boldsymbol{p}\wedge\boldsymbol{\alpha}$<br>$M_0 = -iM_4 = (\boldsymbol{\sigma}\boldsymbol{p})$ | $E\boldsymbol{\sigma}_p + m\boldsymbol{\sigma}_T$<br>$(\boldsymbol{\sigma}\boldsymbol{p}) = (\boldsymbol{\sigma}_p\boldsymbol{p})$ |
| $S_\lambda = -\gamma_5(p_\lambda + im\gamma_\lambda)$ | $\boldsymbol{S} = (\boldsymbol{\sigma}\hat{\boldsymbol{p}})(\boldsymbol{\alpha}\hat{\boldsymbol{p}})\boldsymbol{p} + m\boldsymbol{\sigma}\beta$<br>$S_0 = -iS_4 = (\boldsymbol{\sigma}\hat{\boldsymbol{p}})(\boldsymbol{\alpha}\hat{\boldsymbol{p}})(E - m\beta)$ | $E\boldsymbol{\sigma}_p + m\boldsymbol{\sigma}_T$<br>$(\boldsymbol{\sigma}\boldsymbol{p}) = (\boldsymbol{\sigma}_p\boldsymbol{p})$ |
| $W_{\alpha\beta} = -\gamma_5\gamma_\mu p_\nu\varepsilon_{\mu\nu\alpha\beta}$ | $\boldsymbol{\Omega} = \boldsymbol{\sigma}\beta E + (\boldsymbol{\sigma}\hat{\boldsymbol{p}})(\boldsymbol{\alpha}\hat{\boldsymbol{p}})\beta\boldsymbol{p}$<br>$\boldsymbol{W} = (\boldsymbol{p}\wedge\boldsymbol{\sigma})\beta$<br>$(\Omega_i = \frac{1}{2}\varepsilon_{ijk}W_{jk}, \; W_i = -iW_{i4})$ | $m\boldsymbol{\sigma}_p + E\boldsymbol{\sigma}_T$<br>$\boldsymbol{p}\wedge\boldsymbol{\sigma} = \boldsymbol{p}\wedge\boldsymbol{\sigma}_T$ |

It is particularly interesting to perform, on the last column of Table I, the non-relativistic limit ($E \to m$, $v \to 0$) and the extreme relativistic limit ($m \to 0$, $v \to 1$). We then see that in the non-relativistic limit the conservation of $\boldsymbol{\Omega}$ or of $\boldsymbol{S}$ tells us that $\boldsymbol{\sigma}$, the spin, is conserved. In the extreme relativistic limit, instead, $\boldsymbol{S}$ gives information only on the longitudinal spin $\boldsymbol{\sigma}_p$, while $\boldsymbol{\Omega}$ gives information on the transverse spin $\boldsymbol{\sigma}_T$. Of course the physical interpretation of $S$ and $M$ is the same, since they coincide weakly. Note also that $S_0$ and $\boldsymbol{M}$ are not hermitian; however they obviously are weakly hermitian, and this is sufficient to insure that the non-hermitian parts do not contribute in the last column of Table I.

## 4. – Conservation laws for the interacting field.

We finally come to the consideration of an interacting spinor field, which shall now fulfil the equation

$$i\gamma_\mu p_\mu \psi(x) = \gamma_\mu \partial_\mu \psi(x) = g\psi(x) , \tag{4.1}$$

where $g$ includes the mass and a source term

$$g = m + g' . \tag{4.2}$$

In particular we shall say that the source term is of the scalar, pseudoscalar, vector, pseudovector or tensor type, if $g'$ is respectively of the form:

$$g' = \varphi(x) , \tag{4.3}$$

$$g' = i\gamma_5 \varphi'(x) , \tag{4.4}$$

$$g' = i\gamma_\mu A_\mu(x) , \tag{4.5}$$

$$g' = i\gamma_5 \gamma_\mu A'_\mu(x) , \tag{4.6}$$

$$g' = \sigma_{\mu\nu} t_{\mu\nu}(x) . \tag{4.7}$$

We also admit that the source fields are hermitian. This implies the condition

$$\gamma_4 g^+ \gamma_4 = g \tag{4.8}$$

and the equation

$$-\partial_\mu \overline{\psi}\gamma_\mu = \overline{\psi} g \tag{4.9}$$

for the conjugate field $\overline{\psi} = \psi^+\gamma_4$. From this equation follows the continuity equation for the « electric » spinor current $j_\mu$

$$j_\mu = i\overline{\psi}\gamma_\mu\psi , \qquad \partial_\mu j_\mu = 0 \tag{4.10}$$

and, more generally, the continuity equation for the current

$$J_\mu = i\overline{\psi}\gamma_\mu A\psi , \qquad \partial_\mu J_\mu = 0 \tag{4.11}$$

provided the field $\psi' = A\psi$ fulfils the same equation of motion as the field $\psi$. The conservation of $A$ is therefore implied by this last condition, or, equivalently, by the condition that $A$ commutes with $D - g'$ (in fact it is suf-

ficient that the commutator vanishes « weakly », *i.e.* when applied to a spinor field fulfilling the (non free) equation of motion (4.1)). Remark also that, together with the conservation of $A$, the conservation of $A^2$, $A^3$, etc., is implied ([8]).

Note that we are not considering the most general interaction that may be undergone by a spinor field: for instance eq. (4.1) does not include the case of Fermi interactions. It is however general enough to include Yukawa type interactions and the electromagnetic interaction. This last case is particularly interesting: in fact we shall find that the conservation laws resulting in the free case from the commutativity of the « spin operators » with $D$ may be maintained, for an interacting spinor field, provided the source fields fulfil some conditions in their space-time dependence . Such conservation laws will therefore be relevant in investigating the motion of a Dirac field interacting with a source having an assigned space-time dependence: a model which is of direct physical interest when the source is the electromagnetic field.

As we have seen, a condition sufficient to insure the conservation of the spin-differential operator $A$ is the requirement that $A$ commutes with $D - g'$

$$[A, D - g'] = 0 . \tag{4.12}$$

In the case of the « spin operators » which commute strongly with $D$ this reduces to the condition

$$[A, g'] = 0 . \tag{4.13}$$

Let us investigate in detail what is implied by this condition for the « spin operators » introduced in the preceding sections.

In the case of $\Sigma$, the condition

$$[\Sigma, g'] = [\gamma_5\gamma_4\boldsymbol{\gamma}\boldsymbol{\partial}, g'] = 0 \tag{4.14}$$

is satisfied if the source $g'$ does not depend on the spatial co-ordinates and is of the type

$$g' = \varphi(t) + i\gamma_5\varphi'(t) + i\gamma_4 A_4(t) + i\gamma_5\gamma_4 A_4'(t) . \tag{4.15}$$

In the case of $M$, the condition

$$[M_\lambda, g'] = [\varepsilon_{\mu\nu\varrho\lambda} p_\mu \sigma_{\nu\varrho}, g'] = 0 \tag{4.16}$$

---

([8]) We will not discuss conservation laws of this kind. Note however that their consideration is not necessarily trivial, in view of the observation of the next to the last paragraph of Sect. **3**.

is satisfied for the $M_{\dot\lambda}$ component, provided $g'$ depends only on the co-ordinate $x_{\dot\lambda}$ and is of the type

$$g' = \varphi(x_{\dot\lambda}) + i\gamma_5 \varphi'(x_{\dot\lambda}) + i\gamma_{\dot\lambda} A_{\dot\lambda}(x_{\dot\lambda}) + i\gamma_5\gamma_{\dot\lambda} A_{\dot\lambda}(x_{\dot\lambda}) . \tag{4.17}$$

Here and in the following the dotted indices have an assigned value and no summation is understood over repeated dotted indices.

In the case of $S$, which does not commute strongly with $D$, we have instead to satisfy the condition eq. (4.12), *i.e.* the condition

$$[S_\lambda, D - g'] = 0 . \tag{4.18}$$

This, using eq. (2.11), becomes

$$-2\gamma_5 p_\lambda D - [S_\lambda, g'] = -2\gamma_5 p_\lambda D + [\gamma_5(p_\lambda + im\gamma_\lambda), g'] = 0 . \tag{4.19}$$

This condition is satisfied for the $\dot\lambda$ component, when applied to a field fulfilling the equation of motion (4.1), provided

$$[p_{\dot\lambda}, g'] = 0 , \tag{4.20}$$

$$[\gamma_5, g'] = 2\gamma_5 g' \quad \text{or} \quad \{\gamma_5, g'\} = 0 , \tag{4.21}$$

$$[\gamma_5\gamma_{\dot\lambda}, g'] = 0 . \tag{4.22}$$

$S_{\dot\lambda}$ is therefore conserved provided $g'$ is independent of $x_{\dot\lambda}$

$$\delta_{\dot\lambda} g' = 0 \tag{4.23}$$

and is of the type

$$g' = i\gamma_\mu A_\mu + i\gamma_5\gamma_{\dot\lambda} A'_{\dot\lambda} , \tag{4.24}$$

with

$$A_{\dot\lambda} = 0 . \tag{4.25}$$

Remark that we have now different conditions for $M$ and $S$, although the two operators coincided in the free field case.

Finally, in the case of $W'_{\alpha\beta}$ the condition

$$[W'_{\alpha\beta}, g'] = [\gamma_5(\gamma_\alpha p_\beta - \gamma_\beta p_\alpha), g'] = 0 \tag{4.26}$$

is satisfied for the $W'_{\dot\alpha\dot\beta}$ component, or, equivalently, for the $W_{\dot\gamma\dot\delta}$ component of $W$ (assuming $\dot\alpha$, $\dot\beta$, $\dot\gamma$, $\dot\delta$ different), provided the source $g'$ does not depend

on the co-ordinates $x_\alpha$, $x_{\dot\beta}$ (*i.e.* only depends on $x_{\dot\gamma}$, $x_\delta$), and is of the type

$$g' = \varphi(x_{\dot\gamma}, x_\delta) + i\gamma_\mu A_\mu(x_{\dot\gamma}, x_\delta) + \sigma_{\mu\nu} t_{\mu\nu}(x_{\dot\gamma}, x_\delta) , \tag{4.27}$$

where

$$A_{\dot\alpha} = A_\beta = 0 \tag{4.28}$$

and

$$t_{\dot\alpha\mu} = t_{\dot\beta\mu} = t_{\mu\alpha} = t_{\mu\beta} = 0 . \tag{4.29}$$

Also, if the source does not depend on the three co-ordinates $x_\alpha$, $x_{\dot\beta}$, $x_\gamma$ (*i.e.* depends only on $x_{\dot\delta}$), then the three components $W_{\dot\delta\beta}$, $W_{\alpha\dot\gamma}$, $W_{\dot\beta\gamma}$ are simultaneously conserved, provided $g'$ is of the type

$$g' = \varphi(x_{\dot\delta}) + i\gamma_{\dot\delta} A_{\dot\delta}(x_{\dot\delta}) . \tag{4.30}$$

We may therefore conclude that, if the source fields satisfy some limitations, it does happen that some quantities are still conserved even in the interacting case. The physical meaning of the conserved quantities may in each case be read out of Table I of the preceding section. Some examples of physical interest are discussed in the following section.

## 5. – Conservation of spin in an external electromagnetic field.

Let us now consider some examples of the conservation laws which are still valid for an interacting Dirac field. We concentrate on the physically interesting case of a Dirac particle in the presence of an assigned electromagnetic field.

The equations of motion for the spinor field are

$$i\gamma_\mu p_\mu \psi(x) - m\psi(x) = ie\gamma_\mu A_\mu(x)\psi(x) + \mu\sigma_{\mu\nu} F_{\mu\nu}(x)\psi(x) , \tag{5.1}$$

where $e$ is the electric charge and we also admit the possibility of an anomalous magnetic moment $\mu$. $F_{\mu\nu}(x)$ is now the electromagnetic tensor

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) \tag{5.2}$$

and $A_\mu(x)$ is the vector potential, fulfilling the Lorentz condition

$$\partial_\mu A_\mu(x) = 0 . \tag{5.3}$$

Let us first consider the case of a normal Dirac particle, *i.e.* having no anomalous magnetic moment. The interaction term reduces then to the vector type only, and the results of the preceding section tell us that the conservation of one component of $S_\lambda$ and of $W_{\alpha\beta}$ may still be maintained in the presence of this interaction, provided the source field $A_\mu(x)$ fulfils some limitations. Note that, although a vector coupling is admitted even in the cases of $\Sigma$ and of $M$, the constraints imposed on this field are so stringent, that such a field cannot be taken as a vector potential describing a non vanishing electromagnetic field (because of the Lorentz condition eq. (5.3)).

Let us investigate first the conditions that the electromagnetic field must fulfil, not to destroy the conservation of one component of $S_\lambda$.

$S_0$ is conserved, provided the vector potential $A_\mu$ is time-independent and its fourth component vanishes. Such a vector potential may describe an arbitrary static magnetic field. Reading out of Table I the physical meaning of $S_0$, and since the magnitude of the momentum is constant in the presence of a static magnetic field, we may therefore state the following

THEOREM (5.1): *If a charged Dirac particle moves in an external time-independent magnetic field, the component of its spin along its momentum is conserved.*

For the conservation of $S_3$ we must require instead that the vector potential be independent of the third co-ordinate and that its third component vanishes. Such a vector potential may describe an electric field orthogonal to the third axis and a magnetic field parallel to the third axis, both independent of the third co-ordinate but otherwise essentially arbitrary, the only further limitation being given by the requirement that the vector potential fulfils the Lorentz condition eq. (5.3). A vector potential fulfilling identically that condition is obtained for instance imposing that each of its components be independent of the corresponding co-ordinate, $\partial_\mu A_\mu = 0$. Such a vector potential is still general enough to describe a magnetic field $\boldsymbol{H} = \operatorname{curl} \boldsymbol{A}$ of the form

$$\boldsymbol{H}(x_1, x_2, t) = \boldsymbol{H}'(x_1, t) + \boldsymbol{H}''(x_2, t), \tag{5.4}$$

where $\boldsymbol{H}'$ and $\boldsymbol{H}''$ are completely arbitrary functions, and an electric field $\boldsymbol{E}$ such that

$$\begin{cases} E_1(x_1, x_2, t) = E_1'(x_1, x_2) + E_1''(x_2, t), \\ E_2(x_1, x_2, t) = E_2'(x_1, x_2) + E_2''(x_1, t), \end{cases} \tag{5.5}$$

where $\boldsymbol{E}'$, $\boldsymbol{E}''$ are also completely arbitrary functions of two variables.

Referring again to Table I for the physical interpretation of $\boldsymbol{S}$, we may therefore state the following

THEOREM (5.2): *If a charged Dirac particle, of mass m and energy E, moves in a region where a magnetic field* $\boldsymbol{H}$ *and an electric field* $\boldsymbol{E}$ *exist, the magnetic field*

*pointing in the fixed direction* $\hat{\boldsymbol{n}}$ *and the electric field being orthogonal to* $\hat{\boldsymbol{n}}$, *both* $\boldsymbol{E}$ *and* $\boldsymbol{H}$ *not changing along the* $\hat{\boldsymbol{n}}$ *direction but being otherwise arbitrary (also possibly time-dependent), then the quantity*

$$(E\boldsymbol{\sigma}_p + m\boldsymbol{\sigma}_T)\cdot\hat{\boldsymbol{n}} \tag{5.6}$$

*is conserved, where* $\boldsymbol{\sigma}_p$ *is the longitudinal spin and* $\boldsymbol{\sigma}_T$ *is the spin transverse to the momentum.*

In the non-relativistic limit, this theorem tells us that $\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}$ is conserved, *i.e.* that the component of spin in the direction of the magnetic field and orthogonal to the electric field is conserved.

In the extreme relativistic limit ($m \to 0$, $p \to E$), this theorem tells us that $E\,\boldsymbol{\sigma}_\varrho\cdot\hat{\boldsymbol{n}}$ is conserved. However, in this limit, the chirality $|\boldsymbol{\sigma}_p|$ is also conserved, since in this limit the equations of motion are invariant under the $\gamma_5$ transformation [9], *i.e.* in this limit also $\gamma_5$ is a constant of motion. Therefore what we learn in this limit is only the trivial fact that the component of the momentum parallel to the magnetic field and orthogonal to the electric field is conserved.

Let us now pass to the consideration of the operator $W$. In order that one of its components be still conserved, two components of the vector potential $A$ must vanish identically and the other two must depend only on the corresponding co-ordinates.

Let us consider first the case

$$A_1 = A_2 = 0\,, \tag{5.7}$$

$$A_3 \equiv A_3(x_3, t)\,, \qquad A_0 \equiv A_0(x_3, t)\,. \tag{5.8}$$

Such a vector potential describes an electric field $\boldsymbol{E}$, pointing in the direction of the third axis and indipendent of the co-ordinates $x_1$ and $x_2$

$$E_3(x_3, t) = \partial_t A_3(x_3, t) - \partial_3 A_0(x_3, t)\,. \tag{5.9}$$

The dependence of $E_3$ on $x_3$ and $t$ is again to a large degree arbitrary, since the two functions $A_3(x_3, t)$ and $A_0(x_3, t)$ may be chosen arbitrarily except for the Lorentz condition eq. (5.3), which now becomes

$$\partial_3 A_3(x_3, t) + \partial_t A_0(x_3, t) = 0\,. \tag{5.10}$$

For instance one could always chose for $A_3$ a completely arbitrary function of the time only, and for $A_0$ a completely arbitrary function of the co-ordinate.

---

[9] B. F. Touschek: *Nuovo Cimento*, **5**, 754 and 1281 (1957).

We know, eq. (4.28) and (4.29), that under these conditions the quantity $W'_{12} - W_{34}$ is conserved. Its physical meaning may be read out of Table I, and remembering that the component of $\boldsymbol{p}$ orthogonal to the electric field is also a constant of motion, we are led to the following

THEOREM (5.3): *If a charged Dirac particle is moving in a region where a (possibly time-dependent) electric field exists, which points in a fixed direction and which is constant on all planes orthogonal to that direction, the component of the spin of the particle orthogonal to its momentum and to the direction of the field is conserved.*

Let us now consider the case with

$$A_3 = A_4 = 0 \,, \tag{5.11}$$

$$A_1 \equiv A_1(x_1, x_2) \,, \qquad A_2 \equiv A_2(x_1, x_2) \,. \tag{5.12}$$

Such a vector potential describes now a time-independent magnetic field $\boldsymbol{H}$, pointing in the direction of the third axis and independent of the co-ordinate $x_3$

$$H_3 = \partial_2 A_1(x_1, x_2) - \partial_1 A_2(x_1, x_2) \,. \tag{5.13}$$

The dependence of $H_3$ on $x_1$ and $x_2$ is again to a large degree arbitrary, since the two functions $A_1(x_1, x_2)$, $A_2(x_1, x_2)$ are only restricted by the Lorentz condition

$$\partial_1 A_1 + \partial_2 A_2 = 0 \,. \tag{5.14}$$

For instance one could chose for $A_1$ an arbitrary function of $x_2$ only, and for $A_2$ an arbitrary function of $x_1$ only.

Under these conditions $W'_{34} = W_{12}$ is conserved. Using Table I we are therefore led to the following

THEOREM (5.4): *If a charged Dirac particle of mass $m$ and energy $E$ is moving in a region, where a time-independent magnetic field exists, which points in a fixed direction $\boldsymbol{n}$ and which does not change along that direction, then the quantity*

$$(m\boldsymbol{\sigma}_p + E\boldsymbol{\sigma}_T)\cdot\hat{\boldsymbol{n}} \tag{5.15}$$

*is conserved, where $\boldsymbol{\sigma}_p$ is the longitudinal spin and $\boldsymbol{\sigma}_T$ is the spin transverse to the momentum.*

This theorem may be combined with theorem (5.2) and, using also the fact that the energy $E$ is conserved in the presence of a static magnetic field, we conclude with the following

COROLLARY (5.1): *Under the hypothesis of Theorem* (5.4) *the two quantities* $\boldsymbol{\sigma}_p \cdot \hat{\boldsymbol{n}}$ *and* $\boldsymbol{\sigma}_T \cdot \hat{\boldsymbol{n}}$ *are separately conserved.*

Notice, however, that these two quantities do not commute. Remark also that, under the conditions of Theorem (5.4), also the magnitude of the momentum and its component in the $\hat{\boldsymbol{n}}$ direction are constant, and therefore the conservation of $\boldsymbol{\sigma}_p \cdot \hat{\boldsymbol{n}}$ is equivalent to the conservation of the chirality $|\boldsymbol{\sigma}_p|$.

Let us now relapse the condition $\mu = 0$, *i.e.* consider a Dirac particle possessing an anomalous magnetic moment. The conservation laws of the Theorems (5.1) and (5.2) do not hold any more([10]). As for the conservation laws of the Theorems (5.3) and (5.4), they are instead still valid, provided the condition eq. (4.30) is fulfilled by the electromagnetic field. But eq. (4.29) and (5.2) are just sufficient to guarantee that this condition is always satisfied. We may therefore state the following

COROLLARY (5.2): *The Theorems* (5.3) *and* (5.4) (*but not the Theorems* (5.1) *and* (5.2) *and the Corollary* (5.1)) *are still valid for a* (*charged or neutral*) *Dirac particle having an anomalous magnetic moment.*

## 6. – Additional invariance properties of a free Dirac field: spin co-ordinates operators.

We now want to introduce other operators, which are slightly more complicated than the « spin operators » of the preceding sections, in that they explicitly involve the co-ordinates, but which also commute with the Dirac operator $D$ and are therefore conserved for a free Dirac field.

The operators we want to consider are

$$C = \tfrac{3}{2} i\gamma_5 + x_\varrho M_\varrho \,, \tag{6.1}$$

$$C' = \tfrac{3}{2} i\gamma_5 + x_\varrho S_\varrho \,, \tag{6.2}$$

$$L_\mu = -i\gamma_\mu + x_\alpha W_{\alpha\mu} \,. \tag{6.3}$$

Obviously $C'$ coincides weakly with $C$

$$C' = C + i\gamma_5 x_\varrho \gamma_\varrho D \sim C \,. \tag{6.4}$$

Also, $C$ and $L$ commute with $D$ strongly

$$[C, D] = [L, D] = 0 \,, \tag{6.5}$$

([10]) In fact the breakdown of these conservation laws is the basis of $g - 2$ measurements.

while $C'$ commutes with $D$ only weakly [4]

$$[C', D] = 2\gamma_5(2i - x_\varrho p_\varrho) D \sim 0 . \tag{6.6}$$

Only $L_0$ is strongly hermitian; it is however easy to show that both $C$ and $C'$ are weakly hermitian.

The physical interpretations (in the sense of the third column of Table I) of these operators may be obtained following the method of Section 3. We write down the complete result only for $L_0 = -iL_1 = (1 - \boldsymbol{x}' \boldsymbol{p}\cdot\boldsymbol{\sigma})\beta$, whose physical interpretation is simply $1 + \boldsymbol{x}\wedge\boldsymbol{p}\cdot\boldsymbol{\sigma}$. As for $C$ and $\boldsymbol{L}$, their physical interpretations are simple only in the static ($\boldsymbol{p} \to 0$) limit, where they reduce respectively to $m\boldsymbol{x}\cdot\boldsymbol{\sigma}$ and $-m\boldsymbol{x}\wedge\boldsymbol{\sigma}$.

## 7. – Spin-coordinates conservation laws for an interacting spinor field.

We now proceed to the consideration of an interacting spinor field, and, following the pattern of Section 4, investigate the conditions which the source $g'$ must satisfy to maintain some of the conservation laws of the preceding section.

The conservation of $C$ is maintained, provided

$$[C, g'] = 0 . \tag{7.1}$$

This condition is fulfilled if $g'$ is of the type

$$g' = \varphi + i\gamma_5\varphi' \tag{7.2}$$

and the source fields fulfil the equations

$$\partial_\varrho \varphi = x_\varrho f , \qquad \partial_\varrho \varphi' = x_\varrho f' , \tag{7.3}$$

$f$, $f'$ being arbitrary functions. The conditions of eq. (7.3) would be satisfied if the fields depended on the co-ordinates only through the invariant $x^2$, *i.e.* if they were spherically symmetric around the origin of co-ordinates in space-time.

The conservation of $C'$ would instead require the condition

$$[C', D - g'] = 0 , \tag{7.4}$$

which does not have a simple solution. We may instead define the new operator

$$C'' = \tfrac{3}{2} i\gamma_5 - \gamma_5 x_\varrho (p_\varrho + ig\gamma_\varrho) , \tag{7.5}$$

which coincides with $C'$ in the absence of interaction. Looking now for the conditions which $g'$ must satisfy in order that $C''\psi$ fulfills the same eq. (4.1) as $\psi$, we find by means of a straightforward calculation

$$[g', \gamma_\mu] = 0 , \tag{7.6}$$

$$\partial_\varrho g' = x_\varrho f , \tag{7.7}$$

where $f$ is an arbitrary function. It turns out therefore that the quantity $C''$ (whose physical interpretation is however less simple, since it contains in its definition the source field) is conserved, provided the source field is of the scalar type and is spherically symmetric around the origin of space-time.

Finally we come to the consideration of the more interesting case of the operator $L$, for which we must investigate the condition

$$[L_\mu, g'] = 0 . \tag{7.8}$$

This condition is satisfied for the $\dot{\lambda}$-th component, provided the source is of the type

$$g' = \varphi + i\gamma_{\dot{\lambda}} A_{\dot{\lambda}} \tag{7.9}$$

and the dependence of the source fields on the three co-ordinates different from $x_{\dot{\lambda}}$ is such to satisfy the equations

$$\partial_\mu \varphi = x_\mu f , \qquad \partial_\mu A_{\dot{\lambda}} = x_\mu B_{\dot{\lambda}} , \tag{7.10}$$

with $\mu \neq \dot{\lambda}$, where $f$ and $B_{\dot{\lambda}}$ stand for arbitrary functions. The dependence of the fields on the co-ordinate $x_{\dot{\lambda}}$ remains completely arbitrary.

This last case is more interesting, because from this case we may get again some information about the physically interesting case of a Dirac particle, charged but without anomalous magnetic moment, moving in an assigned electromagnetic field. Note however that, in order to represent the vector potential of an electromagnetic field, the vector field $A$ (which has only the one component $A_{\dot{\lambda}}$ different from zero) must be independent of the $\dot{\lambda}$-th co-ordinate, because of the Lorentz condition eq. (5.3).

In particular, considering the time component $L_0$, we are led to the following (very plausible)

THEOREM (7.1): *If a charged Dirac particle is moving in a central electrostatic field, the component of its spin in the direction of its angular momentum (with respect to the center of symmetry of the field) is constant.*

Another theorem could also be obtained considering the spatial components of $L$; in that case, however, the electromagnetic field would be required to

have a rather peculiar space-time dependence, behaving as a cilindrical electromagnetic wave.

## 8. – Conclusion.

Some conservation laws connected with the spin have been investigated, in the case of an interacting spinor field. The procedure followed has been to define first certain operators, which are conserved in the free case and whose physical meaning, which is connected with spin, has been determined. Then the case of an interacting field has been considered, and the conditions on the source fields necessary to maintain the preceding conservation laws have been found. Special attention has been concentrated on the physically interesting case of a Dirac particle moving in an external electromagnetic field; for this case some theorems have been shown to hold. Generally these theorems have a very simple physical interpretation, and, especially in the non-relativistic limit, they correspond to the conservation laws which would be guessed on the basis of an elementary classical picture.

* * *

It is a pleasure to thank Prof. M. Cini and Prof. B. F. Touschek for their interest in the present work and for valuable suggestions.

## RIASSUNTO

Si considerano gli operatori covarianti che possono servire per la descrizione dello spin di un campo di Dirac e se ne discute il significato fisico, con lo scopo di studiare le corrispondenti leggi di conservazione nel caso di un campo spinoriale interagente. Una particolare attenzione è dedicata al caso di un campo di Dirac in interazione con un campo elettromagnetico assegnato. Si introducono quindi altri operatori, che contengono esplicitamente le coordinate, e che anche commutano con l'operatore di Dirac $D=\gamma_\mu \partial_\mu - m$ e conducono perciò a leggi di conservazione per un campo spinoriale libero. La validità di tali leggi di conservazione nel caso di un campo spinoriale interagente è studiata, tenendo particolarmente presente il caso di interazione con un campo elettromagnetico assegnato.

IL NUOVO CIMENTO Vol. XX, N. 2 16 Aprile 1961

# Non-Local Transformations and Non-Local Conservation Laws for Free Fields.

F. Calogero

*Istituto di Fisica dell'Università - Roma*
*Istituto Nazionale di Fisica Nucleare - Sezione di Roma*

(ricevuto il 25 Gennaio 1961)

**Summary.** — Non-local transformations are introduced, which, when applied to fields obeying homogeneous non-zero mass equations, define new fields obeying massless equations. These transformations are used to extend to the case of massive fields the conservation laws connected with the invariance properties of massless equations.

## Introduction.

It is well known that the equations of motion for free massless fields show more invariance properties than the equations for massive fields. In particular they are invariant under a 15 parameter group (conformal group) (1), as confronted to the 10 parameter group (Poincaré group) pertinent to the non-zero mass case; also, zero mass equations are invariant under « field translations », *i.e.* addition to the field of a constant; and there are finally additional invariance properties peculiar to zero mass fields of particular spin values (*e.g.* the invariance of the neutrino equation under the Touschek transformation) (2).

To these invariance properties correspond in the well known manner conservation laws, which are satisfied by quantities constructed out of the fields and their first derivatives. Therefore in the case of zero mass fields additional

(1) See, for instance, J. A. McLennan, Jr.: *Nuovo Cimento*, **3**, 1360 (1956).
(2) B. Touschek: *Nuovo Cimento*, **5**, 754 (1957).

conserved quantities may be defined, beside the usual ones (momentum, angular momentum, center of mass motion, charge) [3].

The additional conservation laws previously mentioned, however, are not limited to zero mass fields [1]. In this paper we show how they may be extended to the case of massive fields. The generalization is accomplished by first defining new fields, built out of the massive ones by means of non-local transformations [5], and fulfilling zero mass equations. The conserved quantities are then built out of the new fields by applying the zero mass prescriptions.

Only the spin 0 and $\frac{1}{2}$ fields are considered in detail. However all considerations relative to the spin 0 case apply equally well to the spin 1 case. In Section 1 the well known additional conservation laws for massless fields are summarized, and some remarks concerning possible generalizations are added. In Section 2 transformations to « eliminate the mass » are introduced. In Section 3 the new conserved quantities for massive fields are discussed. In this paper the consideration is always limited to free fields.

The metric $x = (\boldsymbol{x}, x_4 = ix_0)$ is used, yielding $x^2 = x_\mu x_\mu = \boldsymbol{x}^2 - x_0^2$: $\psi_{,\mu}(x)$, or simply $q_{,\mu}(x)$, stands for $\partial q(x)/\partial x_\mu$. $\mathrm{d}x$ stands for $\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}x_0$. All integrals are extended to infinity. $\int \mathrm{d}\sigma_\mu j_\mu(x)$ is the usual covariant generalization of $-\int \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\, j_0(x)$. In treating the spinor field hermitian $\gamma$'s will be used, $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$, $\bar{\psi} = \psi^* \gamma_4$.

## 1. – Conservation laws for massless fields.

We now give a brief summary of the conservation laws resulting from the additional invariance properties of the equations of motion for free massless fields. From the differential conservation laws the conserved quantities are

[3] There seems to be no immediate physical interpretation for these additional conserved quantities; see however, for the electromagnetic field, E. Bessel-Hagen: *Math. Ann.*, **84**, 258 (1921). Anyway the physical interpretation of the conserved quantities is not to be sought in the present context, where the consideration will be limited to free fields.

[4] V. I. Ogievetskii and I. V. Polubarinov: *Žurn. Èksp. Teor. Fiz.*, **37**, 470 (1959), *Sowjet Physics J. E. T. P.*, **37** (10), 335 (1960). I am indebted to P. Curtius for the indication of this paper.

[5] A non-local transformation to eliminate the mass was proposed by M. Cini and B. Touschek: *Nuovo Cimento*, **7**, 442 (1958), for the Dirac field. Transformations to the same purpose for the scalar and the Dirac fields were used, in a context similar to the present one, by V. I. Ogievetskiĭ and I. V. Polubarinov, reference [4]. However both the Cini-Touschek transformation and the generalizations of the Cini-Touschek transformation used by Ogievetskiĭ and Polubarinov are essentially non covariant, the space and time variables being treated quite differently. In fact these transformations are isochronous, the non locality being confined to spatial directions; also they are invariant under spatial translations but not under time translations.

derived in the standard manner, which requires the fields to vanish with sufficient rapidity at infinite spacelike distances.

1˙1. *Scalar field* (6). – Field equation:

$$\varphi_{\mu\mu} = 0 \,. \tag{1.1.1}$$

1˙1.1. Scale tranformation:

$$x' = (1+\lambda)x \,, \qquad \lambda \text{ real} \tag{1.1.1.1}$$

$$\varphi'(x') = (1+\lambda)^{-1}\varphi(x) \,. \tag{1.1.1.2}$$

Conservation law

$$\{(\varphi_\mu\varphi_\varrho + \varphi_\varrho\varphi_\mu)x_\varrho - \varphi_\varrho\varphi_\varrho x_\mu + \varphi_\mu\varphi + \varphi\varphi_\mu\}_{,\mu} = s_{\mu,\mu} = 0 \,. \tag{1.1.1.3}$$

Conserved quantity

$$S = \int \mathrm{d}\sigma_\mu s_\mu \,. \tag{1.1.1.4}$$

1˙1.2. Conformal transformation:

$$x'_\mu = (1 - 2\lambda x + \lambda^2 x^2)^{-1}(x_\mu - \lambda_\mu x^2) \,, \quad x_\mu = (1 + 2\lambda x' + \lambda^2 x'^2)^{-1}(x'_\mu + \lambda_\mu x'^2) \,, \tag{1.1.2.1}$$

$$\varphi'(x') = (1 - 2\lambda x + \lambda^2 x^2)\,\varphi(x) \,. \tag{1.1.2.2}$$

Conservation law

$$\{(2x_\lambda x_\varrho - x^2\delta_{\lambda\varrho})(\varphi_\varrho\varphi_\mu + \varphi_\mu\varphi_\varrho - \varphi_\nu\varphi_\nu\delta_{\mu\varrho}) - 2\varphi^2\delta_{\mu\lambda} + \\ + 2x_\lambda(\varphi\varphi_\mu + \varphi_\mu\varphi)\}_{,\mu} = c_{\mu\lambda,\mu} = 0 \,. \tag{1.1.2.3}$$

Conserved quantity

$$C_\lambda = \int \mathrm{d}\sigma_\mu c_{\mu\lambda} \,. \tag{1.1.2.4}$$

1˙1.3. Field translation:

$$\varphi'(x) = \varphi(x) + \varphi_0 \,, \qquad \varphi_0 \text{ constant.} \tag{1.1.3.1}$$

(6) All formulas are written down, for simplicity, for an hermitian scalar field.

Conservation law

(1.1.1) $$\varphi_{\mu\mu} = 0 .$$

Conserved quantity [7]

(1.1.3.2) $$F = \int \mathrm{d}\sigma_\mu \varphi_\mu .$$

1'2. *Spinor field.* – Field equations:

(1.2.1) $$\gamma_\mu \psi_\mu(x) = 0 , \qquad \overline{\psi}_\mu(x)\gamma_\mu = 0 .$$

1'2.1. Scale transformation:

(1.1.1.1) $$x' = (1+\lambda)x , \qquad \lambda \text{ real}$$

(1.2.1.1) $$\psi'(x') = (1+\lambda)^{-\frac{3}{2}} \psi(x) .$$

Conservation law

(1.2.1.2) $$\{x_\varrho \overline{\psi}\gamma_\mu \psi_\varrho\}_{,\mu} = s_{\mu,\mu} = 0 .$$

Conserved quantity

(1.2.1.3) $$S = \int \mathrm{d}\sigma_\mu s_\mu .$$

1.2.2. Conformal transformation:

(1.1.2.1) $$x'_\mu = (1-2\lambda x+\lambda^2 x^2)^{-1}(x_\mu - \lambda_\mu x^2) , \quad x_\mu = (1+2\lambda x' + \lambda^2 x'^2)^{-1}(x'_\mu + \lambda_\mu x'^2) ,$$

(1.2.2.1) $$\psi'(x') = (1-2\lambda x + \lambda^2 x^2)(1-\lambda\cdot\gamma\, x\cdot\gamma)\psi(x) .$$

Conservation law

(1.2.2.2) $$\{(2x_\lambda x_\nu - x^2\delta_{\lambda\nu})(\overline{\psi}\gamma_\mu\psi_\nu - \overline{\psi}_\nu\gamma_\mu\psi + (\mu \leftrightarrow \nu))\}_{,\mu} = c_{\mu\lambda,\mu} = 0 .$$

Conserved quantity

(1.2.2.3) $$C_\lambda = \int \mathrm{d}\sigma_\mu c_{\mu\lambda} .$$

1'2.3. Field translation:

(1.2.3.1) $$\psi'(x) = \psi(x) + \psi_0 , \qquad \psi_0 \text{ constant.}$$

[7] Of course $F$ transforms as the field itself.

Conservation law

(1.2.2) $$\gamma_\mu \psi_\mu(x) = 0 .$$

Conserved quantity ([7])

(1.2.3.2) $$F = \int \mathrm{d}\sigma_\mu \gamma_\mu \psi .$$

1·2.4. Touschek transformation ([2]):

(1.2.4.1) $$\psi'(x) = \exp[i\alpha\gamma_5]\psi(x) , \qquad \overline{\psi}'(x) = \overline{\psi}(x) \exp[i\alpha\gamma_5] , \qquad \alpha \text{ real.}$$

Conservation law

(1.2.4.2) $$i(\overline{\psi}\gamma_\mu\gamma_5\psi)_{,\mu} = t_{\mu,\mu} = 0 .$$

Conserved quantity

(1.2.4.3) $$T = \int \mathrm{d}\sigma_\mu t_\mu .$$

1·3. *Additional remarks.* – Let us also add a few remarks, showing how the above mentioned conservation laws, as well as the more usual ones, may be generalized.

*a*) If the (spinor or scalar) field $\chi(x)$ fulfils the massless free equation, so does the field

(1.3.1) $$\chi^{(n)} = x_{\mu_1} x_{\mu_2} \ldots x_{\mu_n} \chi_{\mu_1\mu_2\ldots\mu_n}(x) .$$

As a consequence, if the « current » $\tilde{\chi} O_\mu \chi$ is conserved, *i.e.* $(\tilde{\chi} O_\mu \chi)_\mu = 0$, also the « current » $\tilde{\chi}^{(n)} O_\mu \chi^{(l)}$ will be conserved, *i.e.* $(\tilde{\chi}^{(n)} O_\mu \chi^{(l)})_\mu = 0$. Here $\tilde{\chi}$ stands for $\chi^*$ or $\overline{\chi}$ in the scalar and spinor case respectively, and $O_\mu$ is a differential and (eventually) spin operator. $O_\mu$ may contain other unsaturated tensor indices besides $\mu$, and in this case the « current » $\tilde{\chi} O_\mu \chi$ would transform as a general tensor.

For instance the conservation of the « electric » spinor current $\overline{\psi}\gamma_\mu\psi$ implies the conservation of the current $\overline{\psi}\gamma_\mu\psi_\varrho x_\varrho$, which is just the conserved current deduced from the invariance of the massless Dirac equation under the scale transformation.

*b*) Since for a classical (unquantized) scalar field $\varphi_{\mu\mu} = 0$ implies $(\varphi^\alpha)_{\mu\mu} = 0$, in the case of a classical field the conservation of $\varphi^* O_\mu \varphi$ implies the conservation of $\varphi^{*\alpha} O_\mu \varphi^\beta$.

*c*) Since $\gamma_\mu \psi_\mu = 0$ implies $\gamma_\mu \gamma_5 \psi_\mu = 0$, the conservation of the current $\overline{\psi} O_\mu \psi$ implies the conservation of the pseudocurrent $\overline{\psi} O_\mu \gamma_5 \psi$ (or $\overline{\psi} \gamma_5 O_\mu \psi$).

For instance, the conservation of the electric current $\overline{\psi} \gamma_\mu \psi$ implies the conservation of the pseudocurrent $\overline{\psi} \gamma_\mu \gamma_5 \psi$ (associated with the invariance of the free massless Dirac equation under Touschek's gauge transformation), the conservation of the « energy momentum tensor » $\overline{\psi} \gamma_\mu \psi_\nu$ implies the conservation of the pseudotensor $\overline{\psi} \gamma_\mu \gamma_5 \psi_\nu$, etc.

*d*) Since the free spinor fields $\psi$ and $\gamma_5 \psi$ obey the same equation of motion as the free scalar field, the conservation of the current $q^* O_\mu q$ implies the conservation of the current $\overline{\psi} O_\mu \psi$ and of the pseudocurrent $\overline{\psi} O_\mu \gamma_5 \psi$. Notice that this property holds for massive free fields as well as for massless fields.

For instance the currents $\overline{\psi} \overleftrightarrow{\partial}_\mu \psi$ and $\overline{\psi} \overleftrightarrow{\partial}_\mu \gamma_5 \psi$ are conserved, the symbol $\overleftrightarrow{\partial}_\mu$ being defined by $A \overleftrightarrow{\partial}_\mu B = A B_\mu - A_\mu B$.

*e*) Since the field

$$\chi_{(n)} = \chi_{\mu_1 \mu_2 \ldots \mu_n} \tag{1.3.2}$$

obeys the same equation of motion as the free (scalar or spinor) field $\chi$, the conservation of the current $\tilde{\chi} O_\mu \chi$ implies the conservation of the tensor $\tilde{\chi}_{(n)} O_\mu \chi_{(l)}$. Also this property holds as well for massive free fields.

For instance, from the conservation of the spinor electric current $\overline{\psi} \gamma_\mu \psi$ we may immediately infer the conservation of the energy momentum tensor $\overline{\psi} \gamma_\mu \psi_\nu$.

## 2. – Non-local transformations.

The more general non-local transformation is defined as follows

$$\chi'(x) = \int K(x, y) \chi(y) \, \mathrm{d}y \,, \tag{2.1}$$

where $\chi(x)$ is a scalar or spinor field, in general massive. In the following we always assume that we deal with convergent integrals; also we suppose that partial integrations may be performed always neglecting surface terms. Finally, wherever we meet multiple integrals, we shall interchange freely the order of integration.

All the transformations to be considered in the following will be covariant in form. This, however, does not mean that the kernel $K(x, y)$ may depend only on the scalar products $x^2$, $y^2$ and $xy$. We admit dependence also on other vectors, introduced in the definition of the transformation. It is of course

true that one can then single out some special frame of reference (for instance that in which a particular vector has most components zero), and that therefore Lorentz invariance is lost. By the statement that the transformation is covariant in form we only mean that the kernel depends only on scalar products. Dependence only on $x^2$, $y^2$ and $xy$ will be instead implied by the more stringent statement that the transformation is Lorentz invariant. Besides, in the case of the spinor field, the kernel will in general also act on the spinor indices, and therefore may also depend on the scalar products $\gamma x$, $\gamma y$. In the following examples of covariant and of Lorentz invariant transformations will be given.

Translation invariance would require $K(x, y)$ to depend only on $x - y$. We consider below separately the translation invariant transformations, and show that they are not suited for our purpose, *i.e.* to eliminate the mass term in the equations of motion. However they allow for a generalization of the usual conservation laws of field theory.

Non-translation invariant transformations, which are capable to eliminate the mass term in the scalar and spinor field equations, will be defined next.

2·1. *Translation invariant transformations.* – Let us consider the transformation

$$\chi'(x) = \int K(x - y)\,\chi(y)\,\mathrm{d}y\,, \tag{2.1.1}$$

and let us further assume that $K(x - y)$ does not act on the spin degrees of freedom (in the spinor case).

It is then easy to prove that the field $\chi'(x)$ obeys the same (free) equation as the field $\chi(x)$. In fact any differentiation acting on the $x$ variable in $K(x - y)$ may be transferred to the $y$ variable (with a change of sign) and then transferred to act on $\chi(y)$ by means of partial integration (with a second change of sign). Therefore from $D\chi = 0$ follows $D\chi' = 0$, where $D$ represents any differential operator. To prove the above statement one chooses respectively $D = \Box^2 - m^2$ in the scalar case, $D = \gamma_\mu \partial_\mu - m$ in the spinor case.

One may remark at this point that since $\chi'(x)$ obeys the same equation as $\chi(x)$, if $C[\chi(x)]$ is a constant of the motion, so will be $C[\chi'(x)] = C\left[\int K(x - y)\,\chi(y)\,\mathrm{d}y\right]$. This shows that the conservation laws corresponding to the invariance properties of the equations of motion and representing the conservation laws of momentum, angular momentum, etc., may be generalized to conservation laws involving non-local currents. In other words from an invariance property of the equations of motion follows not only the continuity of a local current, but also the continuity of a whole class of non-local currents. This corresponds of course to the trivial fact that in the first quantized theory, if the operator $A$ is a constant of motion, so also is $Af(p)$, $f(p)$ being an arbitrary function of the momentum operator $p$.

Note however that, when the consideration is limited to free fields, any transformation having a kernel which is Lorentz invariant and translation invariant reduces itself to multiplication by a constant. This is easily proved going over to momentum space and noticing that the Fourier transform of $K(x-y)$ may depend only on $p^2$, while the equation of motion restricts the Fourier transform of the field $\chi(x)$ to contain only momenta on the mass shell $p^2 = -m^2$. The above proof holds, in the spinor case, also for transformations acting on the spin degrees of freedom, since the only Lorentz invariant way for the $\gamma$ matrices and the vector $p$ to enter in the Fourier transform of the kernel is through the scalar product $\gamma p$, which also reduces to a constant when acting on the Fourier transform of a free field [8].

2·2. *Non-translation invariant transformations.*

2·2.1. Massive scalar field [6]. – Equation of motion

$$\varphi_{\mu\mu}(x) - m^2 \varphi(x) = 0 \,. \tag{2.2.1.1}$$

We shall now define a new field $\varphi'(x)$, such that the above equation of motion for $\varphi(x)$ will imply

$$\varphi'_{\mu\mu}(x) = 0 \,. \tag{2.2.1.2}$$

The new field $\varphi'(x)$ is defined through

$$\varphi'(x) = \int K(x, y; m)\varphi(y)\,\mathrm{d}y = \boldsymbol{K}(m)\varphi(x) \,, \tag{2.2.1.3}$$

where

$$K(x, y; m) = (2\pi)^{-4}\int \mathrm{d}p \exp\left[ip(x-y)\right] \exp\left[iqx\right] f(p^2) \,, \tag{2.2.1.4}$$

and the vector $q$, which is a function of $p$ and of $m$, $q = q(p, m)$, satisfies the equation

$$q^2 + 2pq = m^2 \,, \tag{2.2.1.5}$$

$f(p^2)$ is an arbitrary function, $f(-m^2) = 1$.

Transformations of this type are not invariant under translations, since the kernels do not depend only on $x - y$. Notice however that the non-inva-

[8] The inclusion of a dependence also on the sign of the energy $\varepsilon(p)$ leads to slightly more general transformations, consisting of multiplication of the positive and energy parts separately by (possibly different) constants.

riance is limited to translations in the direction of $q$, the kernels being instead invariant under all translations in directions orthogonal to $q$. The choice of a timelike $q$ (which is not incompatible with the above condition eq. (2.2.1.5)), would therefore allow to define transformations which are invariant under spatial translations. Of these kind are the transformations proposed by CINI and TOUSCHEK and by OGIEVETSKIĬ and POLUBARINOV [5].

To prove eq. (2.2.1.2) it is convenient to pass to the momentum representation

$$\varphi(x) = (2\pi)^{-\frac{3}{2}} \int \mathrm{d}p\, \delta(p^2 + m^2)\theta(p)\{\hat{q}(p) \exp[ipx] - \hat{q}^*(p) \exp[-ipx]\}. \tag{2.2.1.6}$$

Inserting eq. (2.2.1.6) and (2.2.1.4) into eq. (2.2.1.3) we get

$$\varphi'(x) = (2\pi)^{-\frac{3}{2}} \int \mathrm{d}p\, \delta(p^2 + m^2)\theta(p) \cdot \cdot\{\hat{\varphi}(p) \exp[i(p+q)x] + \hat{\varphi}^*(p) \exp[-i(p-\tilde{q})x]\}, \tag{2.2.1.7}$$

where

$$\tilde{q}(p, m) = q(-p, m) \tag{2.2.1.8}$$

and, as a consequence of eq. (2.2.1.5),

$$\tilde{q}^2 - 2p\tilde{q} = m^2. \tag{2.2.1.9}$$

The proof is then completed inserting eq. (2.2.1.7) into eq. (2.2.1.2) and using eq. (2.2.1.5) and (2.2.1.9).

Note that, since $\varphi(x)$ is restricted to satisfy the (free) equation of motion (2.2.1.1), the Fourier transform of $\varphi(x)$ only contains momenta on the $m$ energy shell. As a consequence also the Fourier transform in the $y$ variable of $K(x, y; m)$ is restricted to contain momenta on the $m$ mass shell, the part containing momenta outside the mass shell giving no contribution to the transformation $\boldsymbol{K}(m)$. Therefore we may drop the function $f(p^2)$ in the definition of the kernel $K(x, y; m)$; also, in the definition of $q$, to be discussed below, we shall always use $p^2 = -m^2$, and wherever an arbitrary function of $p^2$ could be inserted we shall only put a constant [9].

---

[9] Note that an arbitrary function $f'(p)$ could also be inserted in the definition of the kernel. The introduction of such functions, which could be covariant in form but not Lorentz invariant, would allow for a generalization of the transformations and, as a consequence, for a generalization of the conserved quantities to be discussed in Sect. 3. We shall not insist here on this generalization, which is analogous to that discussed in subsection (2·1) in connection with the more usual conservation laws of field theory.

The transformation $\boldsymbol{K}(m)$ is not real, and as a consequence $\varphi'(x) = \boldsymbol{K}(m)\, q(x)$ in not hermitian. The transformation $\boldsymbol{K}^*(m)$ (*i.e.* defined by the kernel $K^*(x, y; m)$) is also a suitable transformation [10]. We have

$$K^*(x, y; m) = K(-x, -y; m) \tag{2.2.1.10}$$

and $q(x) = q(-x)$ would therefore imply $q'^*(x) = q'(-x)$. An hermitian field $q'_H(x)$ obeying eq. (2.2.1.2) may of course be defined through $q'_H = q' + q'^* = (\boldsymbol{K} + \boldsymbol{K}^*)\varphi$. For this field $\varphi(x) = \varphi(-x)$ would imply $\varphi'_H(x) = \varphi'_H(-x)$.

A general realization of $q$ is the following

$$q = m\big((a-1)\,\hat{p} + a\hat{t}\big)\,, \tag{2.2.1.11}$$

where $a$ is a real constant. The case $a = 0$ is uninteresting, since it gives $\varphi'(x) = \varphi'(0) = \text{constant}$. $\hat{p}$ is a unit vector in the direction of $p$

$$\hat{p} = (-p^2)^{-\frac{1}{2}} p\,, \qquad \hat{p}^2 = -1\,, \tag{2.2.1.12}$$

and $\hat{t}$ is a unit vector orthogonal to $p$, and therefore spacelike

$$\hat{t}p = 0\,, \qquad \hat{t}^2 = 1\,. \tag{2.2.1.13}$$

A general realization for $\hat{t}$ is

$$\hat{t}_\sigma = (-\varepsilon_{\mu\nu\varrho\sigma'}\varepsilon_{\mu'\nu'\varrho'\sigma'} t_{\mu\nu} t_{\mu'\nu'} p_\varrho p_{\varrho'})^{-\frac{1}{2}} i\varepsilon_{\mu\nu\varrho\sigma} t_{\mu\nu} p_\varrho\,, \tag{2.2.1.14}$$

where $t_{\mu\nu}$ is an arbitrary antisymmetric tensor. Note that the argument of the square root is always positive, as the square of a vector orthogonal to $p$ and therefore spacelike.

A less general realization of $q$, involving only an arbitrary unit vector $(|l^2|)^{-\frac{1}{2}} l$ besides the constant $a$, is achieved by the special choice for $t_{\mu\nu}$

$$it_{\mu\nu} = \varepsilon_{\mu\nu\varrho\tau} l_\varrho p_\sigma\,. \tag{2.2.1.15}$$

[10] An intuitive picture of the meaning of the transformations $\boldsymbol{K}$ and $\boldsymbol{K}^*$ may be obtained if one admits that all the momenta $p$ are restricted to lie on a hyperplane $\pi$ and makes the choice $q = \bar{q}$, where $\bar{q}$ is a fixed vector orthogonal to $\pi$ and of lenght $m$. The transformation $\boldsymbol{K}$ ($\boldsymbol{K}^*$) corresponds then to the addition of the constant momentum $\bar{q}$ ($-\bar{q}$) to all momenta $p$, the orthogonality of $p$ and $\bar{q}$ implying that all vectors $p' = p \pm \bar{q}$ are on the zero mass shell as a consequence of $p$ being on the $m$ mass shell and $\bar{q}^2 = m^2$. This suggests the possible usefulness of this transformation in investigating the relativistic limit of the elastic scattering of massive particles, since in the scattering process the asymptotic momenta are in fact restricted to lie on a hyperplane.

With this ansatz $\hat{t}$ simplifies to

$$\hat{t} = \left(l^2 + (\hat{p}l)^2\right)^{-\frac{1}{2}}\left((\hat{p}l)\,\hat{p} + l\right) . \tag{2.2.1.16}$$

Another realization of $q$, which involves only the arbitrary unit vector $\left(|s^2|\right)^{-\frac{1}{2}}s$, is [11]

$$q = \left(-(ps) \pm ((ps)^2 + m^2s^2)^{\frac{1}{2}}\right)(s^2)^{-1}s , \tag{2.2.1.17}$$

where the sign of the square root is defined so as to give $q(p, 0) = 0$ for any $s$ (spacelike or timelike).

The transformation properties of $q$ under space inversion depend, of course, on the transformation properties of $t_{\mu\nu}$, when the general definition eq. (2.2.1.11) is used, or of $l$, when the ansatz eq. (2.2.1.15) is used, or finally of $s$, when the definition eq. (2.2.1.17) is used. For instance $q$, when defined in terms of $l$, is evidently a polar vector if $l$ is a polar vector, while it is a mixture of polar and axial vector if $l$ is an axial vector. In the following we shall not be interested in the behavior under space inversion.

The transformation $\boldsymbol{K}(m)$ is not Lorentz invariant; in fact to define it completely it was always necessary to assign a vector or tensor.

We shall now introduce other transformations, which are also capable of defining fields obeying to zero mass equations out of fields obeying to non-zero mass equations, and which are Lorentz invariant. These transformations are obtained in an obvious way from the one previously described, *i.e.* integrating away the dependence on the assigned vector or tensor. A suitable kernel is for instance

$$L(x, y; m) = (2\pi)^{-4}\int \mathrm{d}p \exp\left[ip(x - y)\right]\int \mathrm{d}l'\,\mathrm{d}l'' \exp\left[iqx\right]g(l'^2, l''^2, l'l'') , \tag{2.2.1.18}$$

where $q$ depends on the two vectors $l'$, $l''$ through eq. (2.2.1.11) and (2.2.1.14), having put

$$t_{\mu\nu} = l'_\mu l''_\nu - l'_\nu l''_\mu \tag{2.2.1.19}$$

and where $g(l'^2, l''^2, l'l'')$ is normalized in such a way that

$$\int \mathrm{d}l'\,\mathrm{d}l''\, g(l'^2, l''^2, l'l'') = 1 . \tag{2.2.1.20}$$

Note that since $q$ is independent of $l'^2$, $l''^2$, $l'l''$, the arbitrariness in the trans-

[11] This choice, with a timelike $s$, leads to a transformation which is the straightforward covariant generalization of that used in reference [4].

formation resulting from the arbitrariness in the choice of the function $g$ is only apparent. The same applies to a possible dependence of $g$ on $pl'$, $pl''$. However one is always left with the arbitrariness in the choice of $a$ in the definition of $q$ eq. (2.2.1.11). This allows in fact for the choice of an arbitrary (normalized) function of $a$, since any transformation obtained by the preceding one integrating its kernel on the $a$ variable with a normalized weight is also a suitable transformation.

A possible realization of $g(l'^2, l''^2, l'l'')$ is

$$g(l'^2, l''^2, l'l'') = \tilde{g}(l'^2)\tilde{g}(l''^2) \tag{2.2.1.21}$$

and [12]

$$\tilde{g}(l^2) = (l^2 + b^2 - i\varepsilon)^{-n}(l^2)^{k-2}(m^2)^{n-k}\,, \tag{2.2.1.22}$$

$$b^{2(n-k)} = i\pi^2 B(k,\, n-k)(m^2)^{n-k}\,, \tag{2.2.1.23}$$

where $B(k, n)$ is the Gaussian $\beta$-function and $n > k > 0$.

Other Lorentz invariant transformations may be obtained following the same pattern but using as starting point the other realizations of $q$ discussed above. This given for instance

$$L(x, y; m) = (2\pi)^{-4}\int \mathrm{d}p\,\mathrm{d}l\,\exp\,[ip(x-y)]\,\exp\,[iqx]\,\tilde{g}(l^2)\,, \tag{2.2.1.24}$$

where $q$ is defined by eq. (2.2.1.11) and (2.2.1.16), or

$$L(x, y; m) = (2\pi)^{-4}\int \mathrm{d}p\,\mathrm{d}s\,\exp\,[ip(x-y)]\,\exp\,[iqx]h(s^2, ps, m)\,. \tag{2.2.1.25}$$

where $q$ is defined by eq. (2.2.1.17) and $h(s^2, ps, n)$ is a function, arbitrary except for the condition

$$\int \mathrm{d}s\, g(s^2, ps, 0) = 1\,. \tag{2.2.1.26}$$

Note that in this last case $q$ does not depend on $s^2$, but does depend on $ps$. Therefore the arbitrariness in the choice of the $ps$ dependence of the function $h$ results in a real arbitrariness of the transformation, and is the analog of the arbitrariness connected with the choice of $a$ in the preceding cases. We con-

---

[12] J. M. Jauch and F. Rohrlich: *The theory of photons and electrons* (Cambridge Mass., 1955), p. 457.

clude therefore that all these Lorentz invariant transformations contain in their definition an arbitrary function of one variable (*).

All the transformations discussed in this subsection are non local and non invariant under translations. Also we always have [13]

$$q'(0) = q(0) . \tag{2.2.1.27}$$

All these transformations become the identity in the limit of vanishing mass. This follows from

$$q(p, 0) = 0 \tag{2.2.1.28}$$

and the fact that all kernels go over into $\delta$-functions when $q = 0$, $m = 0$.

2·2.2. Massive spinor field. – This subsection will closely follow the pattern of the preceding one. To avoid trivial repetitions we shall therefore skip all comments and remarks which are not peculiar to the spinor case, referring to the preceding subsection for the discussion of all details.

Equations of motion:

$$\gamma_\mu \psi_\mu(x) - m\,\psi(x) = 0 , \qquad \overline{\psi}_\mu(x)\gamma_\mu + m\,\overline{\psi}(x) = 0 . \tag{2.2.2.1}$$

We define a new field $\psi'(x)$, such that the above equation of motion for $\psi(x)$ implies

$$\gamma_\mu \psi'_\mu(x) = 0 , \qquad \overline{\psi}'_\mu(x)\gamma_\mu = 0 . \tag{2.2.2.2}$$

The new field $\psi'(x)$ is defined through

$$\psi'(x) = \int S(x, y; m)\,\psi(y)\,\mathrm{d}y = \boldsymbol{S}(m)\psi(x) , \tag{2.2.2.3}$$

$$\overline{\psi}'(x) = \int \overline{\psi}(y)\,\overline{S}(x, y; m)\,\mathrm{d}y = \overline{\psi}(x)\overline{\boldsymbol{S}}(m) , \tag{2.2.2.4}$$

---

(*) *Note added in proofs.* – A more compact and general way to define a Lorentz invariant kernel is through

$$L(x, y; m_2, m_1) = (2\pi)^{-4}\int \mathrm{d}p\,\mathrm{d}q\,\exp\left[ip(x-y) + iqx\right]\delta(q^2 + 2pq + m_2^2 - m_1^2) f(p, q; m_2, m_1) ,$$

where the function $f$ may depend in an arbitrary way on $pq$, $\varepsilon(p)$ and $\varepsilon(q)$. The transformation defined by this kernel, when applied to a free field of mass $m_1$, defines a field obeying the free equation with mass $m_2$.

[13] Except in the last mentioned case, unless eq. (2.2.1.26) is valid independently of the $m = 0$ condition.

where

$$S(x, y; m) = (2\pi)^{-4}\int \mathrm{d}p \exp[ip(x-y) \exp[iqx] m^{-1}(i\gamma q + m) , \tag{2.2.2.5}$$

$$\bar{S} = \gamma_4 S^* \gamma_4 . \tag{2.2.2.6}$$

The vector $q$ satisfies eq. (2.2.1.5), and may be realized in a number of ways as has been shown in the preceding subsection.

The proof of eq. (2.2.2.2) is completely analogous to the proof given in the preceding subsection, to which we refer. One uses the momentum representation for $\psi(x)$

$$\psi(x) = (2\pi)^{-\frac{3}{2}}\int \mathrm{d}p\, \delta(p^2 + m^2)\theta(p)\cdot \tag{2.2.2.7}$$
$$\cdot\left\{\frac{i\gamma p + m}{2m}\,\hat{\psi}(p) \exp[ipx] + \frac{-i\gamma p + m}{2m}\,\check{\psi}(p) \exp[-ipx]\right\} ,$$

and the fact that $(i\gamma p - m)\,\hat{\psi}(p) = 0$ implies $i\gamma\cdot(p+q)(i\gamma q + m)\hat{\psi}(p) = 0$.

The transformation $\boldsymbol{S}(m)$ is in general not hermitian, *i.e.* when applied to hermitian fields defines non hermitian fields. An hermitian transformation may always be defined as that leading to $\psi'_H = \psi' + \psi'^*$ when applied to $\psi_H = \psi + \psi^*$.

The transformation $\boldsymbol{S}(m)$ is not Lorentz invariant. Lorentz invariant transformations are instead defined, following the notation of eq. (2.2.1.18), (2.2.1.24) and (2.2.1.25) respectively, by the kernels (*)

$$T(x, y; m) = \tag{2.2.2.8}$$
$$= (2\pi)^{-4}\int \mathrm{d}p \exp[ip(x-y)]\int \mathrm{d}l'\,\mathrm{d}l'' \exp[iqx] m^{-1}(i\gamma q + m)\, g(l'^2, l''^2, l'l'') ,$$

or

$$T(x, y; m) = (2\pi)^{-4}\int \mathrm{d}p\,\mathrm{d}l \exp[ip(x-y)] \exp[iqx] m^{-1}(i\gamma q + m)\tilde{g}(l^2), \tag{2.2.2.9}$$

(*) *Note added in proofs.* – A more compact and general way to define a Lorentz invariant kernel is through

$$T(x, y; m_2, m_1) = (2\pi)^{-4}\int \mathrm{d}p\,\mathrm{d}q \exp[ip(x-y) + iqx]\cdot$$
$$\cdot\delta(q^2 + 2pq + m_2^2 - m_1^2)\,\frac{iq\gamma + m_1 + m_2}{2(m_1 + m_2)}\, f(p, q; m_2, m_1) ,$$

where the function $f$, which does not contain $\gamma$-matrices, may depend in an arbitrary way on $pq$, $\varepsilon(p)$, $\varepsilon(q)$. The transformation defined by this kernel, when applied to a free field of mass $m_1$, defines a field obeying the free equation with mass $m_2$.

or finally

$$(2.2.2.10) \qquad T(x, y; m) =$$

$$= (2\pi)^{-4} \int \mathrm{d}p \, \mathrm{d}s \exp\left[ip(x - y)\right] \exp\left[iqx\right] m^{-1}(i\gamma q + m) h(s^2, ps, m) \, .$$

Note that, as is well known, the field $\gamma_5 \psi'(x)$ fulfils the same equation as $\psi'(x)$ does.

All the transformations defined in this subsection are non-local and non-invariant under translations.

To insure that these transformations become the identity in the limit of vanishing mass one should define $q$ in such a way that it goes to zero faster than $m$ when $m$ goes to zero.

### 3. – Conservation laws for massive fields.

In Section **1** we summarized the additional conservation laws which are satisfied by quantities constructed as bilinear combinations of massless fields. All the conservation laws written down were immediate consequences of the massless equations of motion satisfied by the fields.

It is now evident how one can define quantities constructed as bilinear combinations of massive fields, which also fulfil conservation laws analogous to those for massless fields. It is in fact sufficient to use the primed fields instead of the (unprimed) original ones, where the priming refers to the notation used in Section **2**. Since the primed fields obey massless equations of motion, conserved quantities may be obtained in terms of the primed fields by straightforward application of the zero mass prescriptions. To obtain finally the conserved quantities in terms of the original massive fields it suffices to insert in place of the primed fields their definitions in terms of the unprimed ones given in Section **2**. The procedure being completely straightforward we don't think it worth while to write down explicitly the results. We shall rather discuss in detail one example in the next subsection.

The conserved currents ([14]) obtained in this manner are, in general, non-local: *i.e.*, the value of a conserved current at one space-time point depends on the values of the fields (and their first derivatives) over a whole space-time region.

Note however that, in the scalar case, as a consequence of the structure

---

([14]) Here and throughout the words « conserved current » are used as an abbreviation for « current density satisfying a differential conservation law (continuity equation) of the type $j_{\mu,\mu}(x) = 0$ ».

of the transformations defined in Subsection (2·2.1), the non-local character of the conserved currents disappears in the origin of co-ordinates, *i.e.* the value of a conserved current at the origin only depends on the values of the field and its first derivatives at the origin ([13]).

Finally we mention that, since the transformations defined in Section 2 contain in their definition arbitrary functions, from each invariance property one is actually led to the definition not only of one conserved current, but rather of a whole class of (non local) conserved currents. This happens just as well in the case of the more usual conservation laws of field theory, as has been pointed out in Subsection (2·1), as soon as the possibility of making non-local transformations is taken into account.

3·1. *An example.* – We want to discuss one non-local conserved current for massive fields, derived according to the above procedure from a local quantity which is conserved in the case of a massless field. We will concentrate on a case in which the conservation law in the massless case is particularly simple and has been completely elucidated ([15]), *i.e.* the conservation of the pseudocharge $T$ connected with the invariance under Touschek's gauge transformation eq. (1.2.4.1).

In the massive case the conserved quantity is written

$$T = i\int \overline{\psi}'(x)\gamma_\mu\gamma_5\psi''(x)\,\mathrm{d}\sigma_\mu\,, \tag{3.1.1}$$

where the primed fields are obtained applying to the original field $\psi(x)$ one of the transformations previously introduced. Remark that the fields $\psi'$ and $\psi''$ may be different, *i.e.* obtained applying different transformations to the field $\psi$. In fact the only condition required to insure the conservation of $T$ is that both transformations be of the type introduced in Section 2. However, for simplicity, in the following we consider the case with $\psi' = \psi''$. Also, we will limit the consideration to the more simple case of the non Lorentz invariant transformations.

Going over to momentum space, we write

$$\psi(x) = (2\pi)^{-\frac{3}{2}}\int \mathrm{d}\boldsymbol{p}\sum_s \{\hat{a}(\boldsymbol{p}, s)u(p, s)\exp[ipx] + \\ + \breve{a}^*(\boldsymbol{p}, s)v(p, s)\exp[-ipx]\}\,. \tag{3.1.2}$$

In this expression $p_0$ is on the positive mass shell

$$p_0 = +(\boldsymbol{p}^2 + m^2)^{\frac{1}{2}} \tag{3.1.3}$$

([15]) B. Touschek: *Nuovo Cimento*, **5**, 1281 (1957).

and $u(p, s)$, $v(p, s)$ are defined by

$$(3.1.4)\quad \begin{cases} i\gamma p\, u(p,s) = m\, u(p,s)\,; \quad i\gamma_5(\gamma S)\, u(p,s) = s\, u(p,s)\,; \quad \bar{u}(p,s)\, u(p,s') = \delta_{ss'}, \\ i\gamma p\, v(p,s) = -\, m\, v(p,s)\,; \quad i\gamma_5(\gamma S)\, v(p,s) = -\, s\, v(p,s)\,; \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \bar{v}(p,s)\, v(p,s') = -\, \delta_{ss'}\,. \end{cases}$$

$S$ being a spacelike unit fourvector orthogonal to $p$

$$(3.1.5)\qquad S^2 = 1, \qquad (Sp) = 0\,.$$

With this notation we obtain for $T$ (neglecting a $c$-number constant)

$$(3.1.6)\qquad T = \int \mathrm{d}^3\boldsymbol{p} \sum_{ss'} \{\breve{a}(\boldsymbol{p}, s)\, \hat{a}(-\boldsymbol{p}, s')\bar{v}(p, s)\cdot$$
$$\cdot (i\gamma\tilde{q} + m)^2\gamma_4\gamma_5 u(\overset{0}{p}, s')\, f_1(p, q)\, \exp\,[-2i(p_0 - \tilde{q}_0)] +$$
$$+\, \breve{a}^*(\boldsymbol{p}, s)\, \hat{a}^*(-\boldsymbol{p}, s)\bar{u}(\overset{0}{p}, s')(i\gamma\overset{0}{q} + m)^2\gamma_1\gamma_5 v(p, s) f_1(-\overset{0}{p}, \overset{0}{\tilde{q}})\, \exp\,[2i(p_0 - \tilde{q}_0)] +$$
$$+\, \hat{a}^*(\boldsymbol{p}, s)\, \hat{a}(\boldsymbol{p}, s')\bar{u}(p, s)(i\gamma pq + m)\gamma_4\gamma_5(i\gamma q + m) u(p, s') f_2(p, q) +$$
$$+\, \breve{a}^*(\boldsymbol{p}, s)\, \breve{a}(\boldsymbol{p}, s')\bar{v}(p, s')(i\gamma\tilde{q} + m)\gamma_4\gamma_5(i\gamma\tilde{q} + m) v(p, s) f_2(-p, \tilde{q})\}\,,$$

where

$$(3.1.7)\qquad \overset{0}{p} = (-\boldsymbol{p}, ip_0)\,, \qquad \overset{0}{q} = q(\overset{0}{p}, m)$$

and we have assumed

$$(3.1.8)\qquad \tilde{\boldsymbol{q}} = \overset{0}{\boldsymbol{q}}\,, \qquad \tilde{q}_0 = -\,\overset{0}{q}_0\,,$$

$f_1$, $f_2$ are functions of $p$ and are uninteresting for the present discussion.

This expression in general is not diagonal in the particle number, and contains an explicit time dependence. However if we make the special choice (consistent with eq. (2.2.1.5))

$$(3.1.9)\qquad q = mS\,, \qquad \tilde{q} = -\,mS\,.$$

$T$ simplifies to

$$(3.1.10)\qquad T = \int \mathrm{d}\boldsymbol{p} \sum_s \{\hat{a}^*(\boldsymbol{p}, s)\, \hat{a}(\boldsymbol{p}, s)\bar{u}(p, s)\gamma_4(s + \gamma_5) u(p, s) f_1' +$$
$$+\, \breve{a}^*(\boldsymbol{p}, s)\, \breve{a}(\boldsymbol{p}, s)\bar{v}(p, s)\gamma_4(s + \gamma_5) v(p, s) f_2'\}\,.$$

The physical meaning of this quantity becomes quite clear if we make the further choice (consistent with eq. (3.1.5), (3.1.9) and (3.1.8))

$$(3.1.11)\qquad S = (1 - v)^{-\frac{1}{2}}(\hat{\boldsymbol{p}}, iv)\,,$$

where

$$\hat{\boldsymbol{p}} = \boldsymbol{p}/|\boldsymbol{p}|\,, \qquad v = p_0/|\boldsymbol{p}|\,. \tag{3.1.12}$$

With this choice $\hat{a}^*(p, s)$ and $\check{a}^*(p, s)$ become respectively the creation operator for a particle or an antiparticle spinning along (if $s = 1$) or against (if $s = -1$) the direction of motion, while the operator $T$ becomes simply

$$T = \int \mathrm{d}\boldsymbol{p} \sum_s s\{g_1(p)\,\hat{a}^*(p, s)\,\hat{a}(p, s) + g_2(p)\,\check{a}^*(p, s)\,\check{a}\,(p, s)\}\,. \tag{3.1.13}$$

$T$ gives therefore a linear combination of the numbers of particles and antiparticles spinning forward minus the same combination of the numbers of particles and antiparticles spinning backward. The coefficients of the linear combination depend on the momentum and are irrelevant to our discussion.

That this quantity is a constant of motion for a free field is a trivial fact. Our purpose was to show how this constant of motion may be derived from a conservation law peculiar to massless particles. We have also seen that the conservation laws one may obtain with this procedure are more general; however the physical interpretation of the conserved quantities is in general less clear since these quantities are generally not diagonal in the particle number.

## 4. – Conclusion.

Non-local transformations have been introduced which, when applied to free massive fields, define new fields obeying massless equations of motion. All these transformations are covariant in form: however a distinction has been drawn between the transformations which are Lorentz invariant, and the transformations which are not Lorentz invariant, in so far as they contain assigned vectors in their definition. Examples of both types of transformations were given. All these transformations are non-invariant under space-time translations.

These transformations were used to generalize to the case of massive fields the conservation laws peculiar to massless fields. The resulting bilinear conserved quantities turned out to be, in general, non-local, *i.e.* in their definition the fields entered at different space-time points.

One example has been treated in some detail. It has been shown that, although in general the resulting conserved quantities do not have a simple physical interpretation, inasmuch as they are not diagonal in the number of particles, quantities having a simple physical interpretation may be obtained in special cases.

In this paper only the case of free fields has been considered. It is not expected that any of the conservation laws mentioned above is maintained in the case of a realistic theory of interacting fields. However the transformations introduced in Section 2 may provide a clue for a systematic approach to the problem of evaluating the cross-section for a field theoretical process in the extreme relativistic limit. Also, these transformations could represent a general approach to the study of « quasi particle variables » of the type introduced by BOGOLJUBOV in the theory of superconductivity and recently applied by NAMBU to the theory of elementary particles ([16]).

* * *

It is a pleasure to aknowledge an illuminating discussion on the invariance properties of massless fields with E. CORINALDESI, and an interesting conversation with G. JONA-LASINIO. The author wishes also to thank the University of Grenoble, the present research being initiated while he was attending the Summer School at Les Houches, H.S., France.

([16]) N. N. BOGOLJUBOV, V. V. TOLMACHEV and A. V. SHIRKOV: *A New Method in the Theory of Superconductivity* (New York, 1959); Y. NAMBU and G. JONA-LASINIO *A Dynamical Model of Elementary Particles based on an Analogy with Superconductivity* ((University of Chicago preprint, to be published in the *Phys. Rev.*).

## RIASSUNTO

Si introducono delle trasformazioni non locali che, se applicate a campi che obbediscono a equazioni con massa diversa da zero, definiscono nuovi campi che obbediscono a equazioni con massa uguale a zero. Queste trasformazioni sono utilizzate per estendere al caso dei campi con massa le leggi di conservazione connesse alle proprietà di invarianza delle equazioni dei campi senza massa.

# Transient Effects in Nuclear Magnetic Resonance with the Rotating Co-ordinates Method.

G. Bonera and P. De Stefano

*Istituto di Fisica dell'Università - Pavia*

(ricevuto il 28 Gennaio 1961)

**Summary.** — Some transient effects in nuclear magnetic resonance are studied with the rotating coordinates method. A description of the motion of the nuclear magnetization during the transition is given by means of cinematic models.

## 1. – Introduction.

It is known that many problems regarding nuclear magnetic resonance can be solved more easily by use of the rotating co-ordinates system introduced by Raby, Ramsey and Schwinger (1).

Recently Bonera and Giulotto (2) have applied the rotating co-ordinates method to a problem in which the effects of the oscillating magnetic field and those of the relaxation must be considered at the same time. More precisely these authors have studied the motion of the nuclear magnetization during a slow adiabatic passage and have given a simple geometrical picture of it.

In the present paper we want to investigate, using some results of ref. (2), how the nuclear magnetization, due to a static magnetic field, approaches its equilibrium value when the rotating field is applied suddenly.

The transient effects which take place under these conditions have already been studied by Torrey (3) starting from Bloch's equations (4). Our treatment,

(1) I. I. Raby, N. F. Ramsey and J. Schwinger: *Rev. Mod. Phys.*, **26**, 167 (1954).
(2) G. Bonera and L. Giulotto: *Nuovo Cimento*, **14**, 435 (1959).
(3) H. C. Torrey: *Phys. Rev.*, **76**, 1059 (1949).
(4) F. Bloch: *Phys. Rev.*, **70**, 460 (1946).

which makes use of the rotating co-ordinates method, has the advantage of being analytically simpler and of allowing a description of the motion of the nuclear magnetization during the transition by a simple cinematic model.

## 2. – Equation of motion of the magnetization for the rotating co-ordinates system.

Let us consider an assembly of identical nuclei with magnetogyric ratio $\gamma$ in a steady magnetic field $\boldsymbol{H}_0$ and let $\boldsymbol{M}_0 = \chi \boldsymbol{H}_0$ be the nuclear magnetization $\boldsymbol{M}$ in thermal equilibrium.

We apply a magnetic field $\boldsymbol{H}_1$ rotating with angular speed $\omega$ in a plane perpendicular to $\boldsymbol{H}_0$. In order to study the effect of this field, let us refer to a system of axes $x'y'z'$ rotating with the same speed $\omega$ of $\boldsymbol{H}_1$ around the $z'$ axis directed along $\boldsymbol{H}_0$. The effective magnetic field for this reference system is $\boldsymbol{H}_{\text{eff}} = \boldsymbol{H}_1 + (\boldsymbol{H}_0 + \boldsymbol{\omega}/\gamma)$.

It is known that the effect of the rotating field, for an observer that makes use of the reference axes $x'y'z'$, consists, in absence of relaxation, in a precession of $\boldsymbol{M}$ around $\boldsymbol{H}_{\text{eff}}$ with angular speed $\omega_r = |\gamma| H_{\text{eff}}$. If we account for the relaxation too, we can write the equation of motion of $\boldsymbol{M}$ in the form

$$\frac{\mathrm{d}\boldsymbol{M}}{\mathrm{d}t} = \left(\frac{\mathrm{d}\boldsymbol{M}}{\mathrm{d}t}\right)_{\text{rot}} + \left(\frac{\mathrm{d}\boldsymbol{M}}{\mathrm{d}t}\right)_{\text{relax}}, \tag{1}$$

where $(\mathrm{d}\boldsymbol{M}/\mathrm{d}t)_{\text{rot}}$ represents the speed of the variation of $\boldsymbol{M}$ due to the precession around $\boldsymbol{H}_{\text{eff}}$ and $(\mathrm{d}\boldsymbol{M}/\mathrm{d}t)_{\text{relax}}$ represents the speed of the variation of $\boldsymbol{M}$ due to the relaxation.

Eq. (1) represents a precessional motion perturbed by the relaxation; this motion will bring $\boldsymbol{M}$, after a long enough time, to an equilibrium position characterized by the equation

$$\frac{\mathrm{d}\boldsymbol{M}}{\mathrm{d}t} = \left(\frac{\mathrm{d}\boldsymbol{M}}{\mathrm{d}t}\right)_{\text{rot}} + \left(\frac{\mathrm{d}\boldsymbol{M}}{\mathrm{d}t}\right)_{\text{relax}} = 0 .$$

The solution of this equation has been given in ref. [2]; at the equilibrium the extremity of $\boldsymbol{M}$ lies on the equilibrium ellipsoid and its position on this ellipsoid depends on the intensity of the field $\boldsymbol{H}_1$ and on the difference $\omega - \omega_0$ between the frequency of $\boldsymbol{H}_1$ and the Larmor frequency relative to $\boldsymbol{H}_0$.

## 3. – Transient effects.

Here we want to solve the equation of motion (1) and to study how the nuclear magnetization $\boldsymbol{M}$, after the application of the field $\boldsymbol{H}_1$, reaches its equilibrium position. As in ref. [2], we shall characterize the vector $\boldsymbol{M}$ by

giving its modulus $M$, the angle $\Omega$ that the plane determined by $\boldsymbol{M}$ and $\boldsymbol{H}_{\text{eff}}$ forms with the plane $x'z'$ and the angle $\alpha$ that $\boldsymbol{M}$ forms with $\boldsymbol{H}_{\text{eff}}$.

Let $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ be three unit vectors which characterize the direction of $\boldsymbol{M}$, the direction perpendicular to $\boldsymbol{M}$ in the plane of $\boldsymbol{M}$ and $\boldsymbol{H}_{\text{eff}}$ and the direction perpendicular to this plane, respectively. Eq. (1) is equivalent to the following system of three scalar equations:

$$(2a) \qquad \frac{\mathrm{d}M}{\mathrm{d}t} = \left(\frac{\mathrm{d}M_{x'}}{\mathrm{d}t}\right)_{\text{relax}} A_{x'} + \left(\frac{\mathrm{d}M_{y'}}{\mathrm{d}t}\right)_{\text{relax}} A_{y'} + \left(\frac{\mathrm{d}M_{z'}}{\mathrm{d}t}\right)_{\text{relax}} A_{z'} ,$$

$$(2b) \qquad -M\frac{\mathrm{d}\alpha}{\mathrm{d}t} = \left(\frac{\mathrm{d}M_{x'}}{\mathrm{d}t}\right)_{\text{relax}} B_{x'} + \left(\frac{\mathrm{d}M_{y'}}{\mathrm{d}t}\right)_{\text{relax}} B_{y'} + \left(\frac{\mathrm{d}M_{z'}}{\mathrm{d}t}\right)_{\text{relax}} B_{z'} ,$$

$$(2c) \qquad M\sin\alpha\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \left(\frac{\mathrm{d}M_{x'}}{\mathrm{d}t}\right)_{\text{relax}} C_{x'} + \left(\frac{\mathrm{d}M_{y'}}{\mathrm{d}t}\right)_{\text{relax}} C_{y'} + \left(\frac{\mathrm{d}M_{z'}}{\mathrm{d}t}\right)_{\text{relax}} C_{z'} + M\omega_r\sin\alpha .$$

An interesting solution, which can be represented with a simple picture, is obtained under the assumption (physically correct in several cases) that the period of precession of $\boldsymbol{M}$ around $\boldsymbol{H}_{\text{eff}}$ is small compared with the relaxation times $T_1$ and $T_2$, *i.e.*, supposing $T_1 \geqslant T_2$,

$$(3) \qquad \omega_r = |\gamma|\,(H_1^2 + (H_0 - \omega/\gamma)^2)^{\frac{1}{2}} \gg \frac{1}{T_2} .$$

Under this condition we can neglect the terms due to the relaxation in eq. (2*c*) and we write simply

$$M\sin\alpha\frac{\mathrm{d}\Omega}{\mathrm{d}t} = M\omega_r\sin\alpha ,$$

*i.e.* the plane of $\boldsymbol{M}$ and $\boldsymbol{H}_{\text{eff}}$ rotates around $\boldsymbol{H}_{\text{eff}}$ with constant angular speed $\omega_r$. Then the motion of $\boldsymbol{M}$ can be studied in this plane.

We can take as a reference system on this plane two axes directed along $\boldsymbol{H}_{\text{eff}}$ and along the perpendicular to $\boldsymbol{H}_{\text{eff}}$, respectively. In this reference system the eq. (2*a*) and (2*b*) can be written as follows (if we put the relaxation terms in the explicit form given by BLOCH and call $\theta$ the angle formed by $\boldsymbol{H}_{\text{eff}}$ with the $z'$ axis):

$$(4) \qquad \begin{cases} \dfrac{\mathrm{d}M_{\parallel}}{\mathrm{d}t} = -M_{\parallel}\left(\dfrac{\sin^2\theta}{T_2} + \dfrac{\cos^2\theta}{T_1}\right) + \dfrac{M_0\cos\theta}{T_1} , \\ \dfrac{\mathrm{d}M_{\perp}}{\mathrm{d}t} = -M_{\perp}\dfrac{1}{2}\left(\dfrac{1}{T_2} + \dfrac{\sin^2\theta}{T_1} + \dfrac{\cos^2\theta}{T_2}\right); \end{cases}$$

here $M_{\parallel}$ and $M_{\perp}$ are the components of $\boldsymbol{M}$ along $\boldsymbol{H}_{\text{eff}}$ and along the perpendicular to $\boldsymbol{H}_{\text{eff}}$, respectively.

The solution of the system (4) is:

$$(5)\qquad \begin{cases} M_{\parallel}(t) = \left[M_{\parallel}(0) - \dfrac{M_0 \cos\theta}{(T_1/T_2)\sin^2\theta + \cos^2\theta}\right] \exp\left[-t/T_{\parallel}\right] + \\ \qquad\qquad + \dfrac{M_0 \cos\theta}{(T_1/T_2)\sin^2\theta + \cos^2\theta}\,, \\ M_{\perp}(t) = M_{\perp}(0)\exp\left[-t/T_{\perp}\right)\,, \end{cases}$$

where

$$\frac{1}{T_{\parallel}} = \frac{\sin^2\theta}{T_2} + \frac{\cos^2\theta}{T_1}\,,$$

$$\frac{1}{T_{\perp}} = \frac{1}{2}\left(\frac{1}{T_2} + \frac{\sin^2\theta}{T_1} + \frac{\cos^2\theta}{T_2}\right)$$

and $M_{\parallel}(0)$ and $M_{\perp}(0)$ are the values of the components of $\boldsymbol{M}$ in the initial conditions.

We assume at first $T_1 = T_2 = T$: then the eq. (5) become:

$$(6)\qquad \begin{cases} M_{\parallel}(t) = \left(M_{\parallel}(0) - M_0\cos\theta\right)\exp\left[-t/T\right] + M_0\cos\theta\,, \\ M_{\perp}(t) = M_{\perp}(0)\exp\left[-t/T\right]. \end{cases}$$

In the plane of $\boldsymbol{M}$ and $\boldsymbol{H}_{eff}$ the extremity $P$ of $\boldsymbol{M}$ tends to the equilibrium position $P_e$ (intersection of $\boldsymbol{H}_{eff}$ with the equilibrium sphere), moving on a straight line with a speed that decreases with a time constant $T$. Since this plane rotates around $\boldsymbol{H}_{eff}$, the path of the extremity $P$ of $\boldsymbol{M}$, in the rotating reference system, will be a kind of spiral traced on a cone.

If initially we have $\boldsymbol{M} = \boldsymbol{M}_0$, the component $M_{\parallel}$ has already its equilibrium value $M_{\parallel} = M_0\cos\theta$. Then in its motion the extremity $P$ stays on a plane perpendicular to $\boldsymbol{H}_{eff}$ and describes a spiral on this plane (*) (Fig. 1).

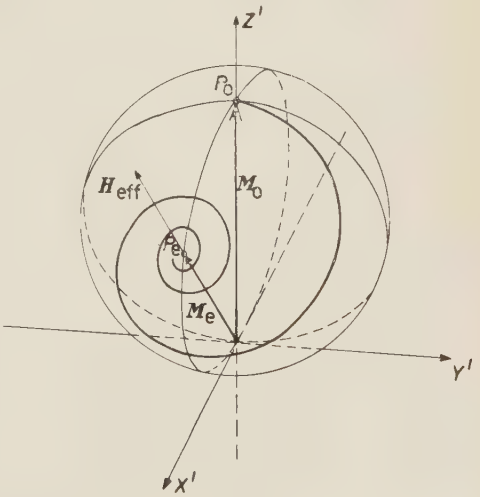


Fig. 1. – The spiral described by the extremity of $\boldsymbol{M}$ on the plane perpendicular to $\boldsymbol{H}_{eff}$ which contains the extremity $P_0$ of $\boldsymbol{M}_0$, when $T_1 = T_2$.

(*) In a system of polar coordinates $\varrho$ and $\Phi$ with the polar axis directed from $P_e$ to $P_0$, the equation of the spiral is $\varrho = \varrho_0 \exp\left(-\Phi/\omega_r T\right)$.

If $T_1 \neq T_2$, the motion of the extremity of $\boldsymbol{M}$, in the plane of $\boldsymbol{M}$ and $\boldsymbol{H}_{eff}$, is in general no longer rectilinear. In this plane the path of $P$ is a curve of the type of those shown in Fig. 2; with reference to the rotating axes, the path is a kind of spiral described on the surface obtained by rotating this curve around $\boldsymbol{H}_{eff}$.

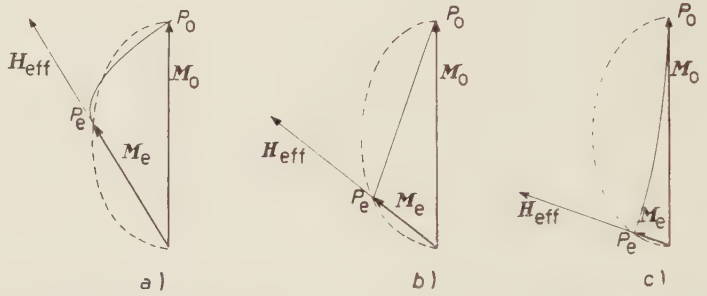


Fig. 2. – In the case $T_1 \neq T_2$ the path of the nuclear magnetization in the plane of $\boldsymbol{M}$ and $\boldsymbol{H}_{eff}$ depends on the angle $\theta$ formed by $\boldsymbol{H}_{eff}$ and $\boldsymbol{H}_0$, *i.e.*, as $\operatorname{tg}\theta = \gamma H_1/(\gamma H_0 - \omega)$, on the magnitude of $\omega$ and $H_1$. Assuming $\boldsymbol{M} = \boldsymbol{M}_0$ initially, we have a path of type *a* for $|\operatorname{tg}\theta| < \sqrt{2}$, of type *b* for $|\operatorname{tg}\theta| = \sqrt{2}$ and of type *c* for $|\operatorname{tg}\theta| > \sqrt{2}$. If $T_1 = T_2$ the path of $P$ is always rectilinear.

If initially $\boldsymbol{M}$ is directed along $\boldsymbol{H}_{eff}$, we see from the eq. (5) that $\boldsymbol{M}$ retains its direction while its modulus tends exponentially to the equilibrium value

$$M_e = \frac{M_0 \cos\theta}{(T_1/T_2)\sin^2\theta + \cos^2\theta} .$$

The time constant is (*)

$$\frac{1}{T} = \frac{\sin^2\theta}{T_2} + \frac{\cos^2\theta}{T_1} .$$

If $|\gamma| H_1 T_2 \gg 1$, the condition (3) certainly holds and the models we have found are valuable whatever is the frequency of the rotating field. If the condition $|\gamma| H_1 T_2 \gg 1$ is not verified these models are valuable only when the difference $|H_0 - \omega/\gamma|$ is large, that is when we are far enough from the resonance, while, when we approach the resonance, the motion of the magnetization becomes so complex that its representation by way of a model is no longer advisable.

(*) This case has already been considered by G. Bonera, L. Chiodi, L. Giulotto and G. Lanzi: *Nuovo Cimento*, **14**, 119 (1959).

However when the frequency of the rotating field is exactly equal to the resonance frequency, a simple model is still possible whatever is the value of $|\gamma| H_1 T_2$.

In fact we have in resonance $\theta = \pi/2$, *i.e.* $\boldsymbol{H}_{\text{eff}}$ is directed along $\boldsymbol{H}_1$. Eq. (2*b*) therefore shows that, if $\boldsymbol{M}$ is initially directed along $\boldsymbol{H}_0$ or if it lies in the $z'y'$ plane (*), we have $\alpha = \pi/2 = \text{const}$. Hence we see that the motion of $\boldsymbol{M}$ is a plane motion in the $z'y'$ plane.

In order to study the motion of $\boldsymbol{M}$ in this plane we remark that we know already from ref. (2) the equilibrium position in resonance $\boldsymbol{M}_e^*$ to which $\boldsymbol{M}$ tends: $\boldsymbol{M}$ tends to lay itself along a direction that forms an angle $\Omega^*$ with $z'$, being $\operatorname{tg} \Omega^* = |\gamma| H_1 T_2$. It will then be convenient to refer the motion of $M$ to two orthogonal axes $\xi$ and $\eta$, directed along $\boldsymbol{M}_e^*$ and the perpendicular to $\boldsymbol{M}_e^*$ in the $z'y'$ plane respectively.

When the eq. (2*a*) and (2*b*) are projected on these axes they become

$$(7)\quad \begin{cases} \dfrac{\mathrm{d}M_\xi}{\mathrm{d}t} = M_\eta \left( \left( \dfrac{1}{T_2} - \dfrac{1}{T_1} \right) \sin \Omega^* \cos \Omega^* + \omega_r \right) - \\ \qquad\qquad - M_\xi \left( \dfrac{\cos^2 \Omega^*}{T_1} + \dfrac{\sin^2 \Omega^*}{T_2} \right) + \dfrac{M_0 \cos \Omega^*}{T_1} , \\ \dfrac{\mathrm{d}M_\eta}{\mathrm{d}t} = M_\xi \left( \left( \dfrac{1}{T_2} - \dfrac{1}{T_1} \right) \sin \Omega^* \cos \Omega^* - \omega_r \right) - \\ \qquad\qquad - M_\eta \left( \dfrac{\cos^2 \Omega^*}{T_2} + \dfrac{\sin^2 \Omega^*}{T_1} \right) + \dfrac{M_0 \sin \Omega^*}{T_1} . \end{cases}$$

Let us consider, at first, the case $T_1 = T_2 = T$; integrating the eq. (7), we obtain, if $\boldsymbol{M}$ has initially the value $\boldsymbol{M}_0$,

$$(8)\quad \begin{cases} M_\xi = M_0 \sin \Omega^* \exp [-t/T] \sin \omega_r t + M_0 \cos \Omega^* , \\ M_\eta = M_0 \sin \Omega^* \exp [-t/T] \cos \omega_r t . \end{cases}$$

These equations tell us that $\boldsymbol{M}$ can be imagined as the sum of two vectors: one $\boldsymbol{M}_e^*$ is fixed along the $\xi$ axis and has the extremity on the circle inter-

(*) If $\boldsymbol{M}$ does not lie initially in this plane, we can always resolve it in two vectors, one directed along $\boldsymbol{H}_{\text{eff}}$ and the other lying on the $z'y'$ plane: the former is subjected to the transversal relaxation only and will tend exponentially to zero with a time constant $T_2$, the latter will move as determined by the relaxation and the precession in a way that we shall find out.

section of the $z'y'$ plane with the equilibrium sphere, the other $\boldsymbol{M}_r$ rotates with angular speed $\omega_r$ around the extremity $P_e^*$ of $\boldsymbol{M}_e^*$ while its modulus tends exponentially to zero with a time constant $T$ (Fig. 3).

In the general case $T_1 \neq T_2$, putting $\delta = \frac{1}{2}|1/T_2 - 1/T_1|$, we distinguish two cases:

*a*) If $\omega_r > \delta$, the solution of the eq. (7) has the following form

$$(8') \qquad \begin{cases} M_\xi = A \exp[-t/T] \cos(\beta t - \varphi) + \dfrac{M_0 \cos \Omega^*}{(T_1/T_2)\sin^2 \Omega^* + \cos^2 \Omega^*}, \\ M_\eta = B \exp[-t/T] \cos(\beta t - \varphi'), \end{cases}$$

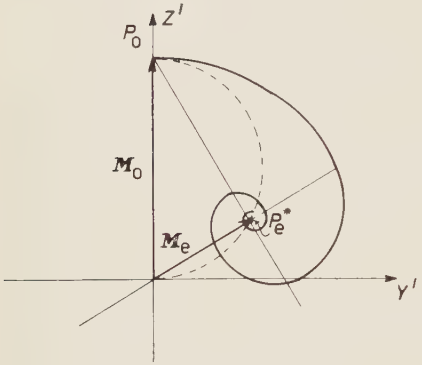


Fig. 3. – The spiral described by the nuclear magnetization on the plane $y'z'$ in resonance.

where

$$\frac{1}{T} = \frac{1}{2}\left(\frac{1}{T_1} + \frac{1}{T_2}\right),$$

$$\beta = (\omega_r^2 - \delta^2)^{\frac{1}{2}}.$$

Constants $A$, $B$, $\varphi$ and $\varphi'$ depend on the initial conditions.

We shall no longer have $\varphi - \varphi' = \pi/2$, as in the case $T_1 = T_2$. The motion is however similar to that seen in this case: the rotating vector $\boldsymbol{M}_r$ rotates with constant angular speed and its modulus oscillates with frequency $\beta/2\pi$ around a mean value that tends exponentially to zero with a time constant $T$.

*b*) If $\omega_r \leqslant \delta$, then the solution of the eq. (7) is

$$(8'') \qquad \begin{cases} M_\xi = A \exp\left[-t\left(\dfrac{1}{T} + (\delta^2 - \omega_r^2)^{\frac{1}{2}}\right)\right] + B \exp\left[-t\left(\dfrac{1}{T} - (\delta^2 - \omega_r^2)^{\frac{1}{2}}\right)\right], \\ M_\eta = A' \exp\left[-t\left(\dfrac{1}{T} + (\delta^2 - \omega_r^2)^{\frac{1}{2}}\right)\right] + B' \exp\left[-t\left(\dfrac{1}{T} - (\delta^2 - \omega_r^2)^{\frac{1}{2}}\right)\right], \end{cases}$$

where $A$, $B$ and $A'$, $B'$ are constants to be determined from the initial conditions. The extremity of $\boldsymbol{M}$ describes a short curve tending with decreasing speed to the equilibrium position on the extremity of $\boldsymbol{M}_e^*$.

In the case $\omega_r = \delta$, the path of the extremity of $\boldsymbol{M}$ is a straight line and the speed decreases exponentially with a time constant $T$.

From the results we have found in the various cases one can easily obtain the analytical expressions for the nuclear signals found by TORREY: it is sufficient to pass from the motion of the vector $\boldsymbol{M}$ to the equations for its components along the $x'$ and $y'$ axes of the rotating co-ordinates system.

* * *

The authors are grateful to Professor L. GIULOTTO for his valuable suggestions.

The authors also wish to express their gratefulness to the Consiglio Nazionale delle Ricerche for its grant.

---

## RIASSUNTO

Sono stati analizzati con il metodo delle coordinate rotanti alcuni effetti transienti in risonanza magnetica nucleare. È stata data inoltre per mezzo di modelli cinematici una descrizione del moto della magnetizzazione nucleare durante la transizione.

IL NUOVO CIMENTO Vol. XX, N. 2 16 Aprile 1960

# On the Observability of the Signs of the Strong Interaction Coupling Constants (*).

D. B. Lichtenberg

*Physics Department Michigan State University - East Lansing, Mich.*

(ricevuto il 2 Febbraio 1961)

**Summary.** — Criteria are given for the observability of the signs of coupling constants appearing in elementary particle interactions. The results are applied to particular strong interaction Lagrangians. For a commonly considered meson-baryon interaction with eight coupling constants, it is shown that only four (independent) relative signs are observable. The signs of the coupling constants in certain meson-meson interactions are discussed briefly. Some difficulties in measuring the observable signs are mentioned.

## 1. – Introduction.

A Lagrangian describing the interaction of two elementary particles generally contains a parameter which is a measure of the strength of the interaction, the so-called coupling constant. The magnitude of this coupling constant is an observable; indeed it must be determined from experiment, since in current theory there is no way to predict it.

However, the phase of the coupling constant is not necessarily an observable. For example, consider the familiar case of the coupling constant in the electromagnetic interaction, the electric charge of a particle. This charge is in general taken to be real. However, it need not be, since if it is multiplied by an arbitrary phase and at the same time the electromagnetic field is multiplied by the opposite phase, the interaction is unchanged. This means that this phase is not an observable. It is convenient to *define* the electromagnetic

(*) Work supported in part by the National Science Foundation.

field to be real. Once this is done, it is meaningful to ask whether the coupling constant is real. The experimental observation that the electromagnetic interaction is invariant under charge conjugation then shows that the electric charge must be a real number.

If the coupling constant is real, we can further ask whether it is positive or negative. Again it is apparent that the sign of the electric charge of a single particle is not an observable, since a sign change is just a special case of a phase transformation. However, the relative sign of the charges of two particles is an observable, since there is no transformation which can alter such a relative sign and still leave the interaction invariant. In such a case, it is convenient to *define* the charge of one particle to be positive (or negative) and then determine experimentally the signs of the charges of other particles. Thus, as is well known, the electron is defined to have negative charge, and within this convention the signs of the charges of other particles are determined by experiment.

In this paper we are primarily concerned with the observability of the signs of the coupling constants governing the strong interactions, but our considerations will apply to other interactions as well.

In order to determine whether the signs of the strong interaction coupling constants are observable, we must make some assumption about the form of the interaction Lagrangian. We shall assume Lagrangians which are bilinear in the fields of fermions and linear, quadratic or quartic in the fields of bosons. The generalization to Lagrangians containing higher powers of fields offers no difficulty.

In Section 2, the pion-nucleon interaction will be used as an example to illustrate general rules for the observability of coupling constants signs. In Section 3 we apply the rules to a meson-baryon interaction and also briefly consider a possible meson-meson interaction. In Section 4 we discuss the validity of the Lagrangian we have assumed and also briefly treat the question of how one would actually measure each observable coupling constant sign.

## 2. – Coupling constants in the pion-nucleon interaction.

We shall obtain here general criteria for the observability of coupling constant signs by considering a pion-nucleon interaction as an example. To be specific, we pick a Lagrangian which is linear in the pion field and bilinear in the fields of the proton and neutron.

The interaction Lagrangian, which has been considered by many authors [1],

---

[1] See *e.g.*, *Suppl. Prog. Theor. Phys.* (*Japan*), no. 3 (1956), for a list of references.

is of the form

$$L = G_1 \bar{p}\gamma_5 p\pi^0 + G_2 \bar{n}\gamma_5 n\pi^0 + G_3 \bar{p}\gamma_5 n\pi^+ + G_3 \bar{n}\gamma_5 p\pi^- , \tag{1}$$

where the symbol for a particle denotes the field operator which annihilates it, $\gamma_5$ is the pseudoscalar Dirac matrix, and the $G_i$ $(i = 1, 2, 3)$ are coupling constants. In writing (1), we have assumed that parity, charge conjugation and time reversal invariance separately hold, so that the $G_i$ may be taken to be real. The same coupling constant $G_3$ multiplies the third and fourth terms in the Lagrangian to make it Hermitian. This Lagrangian is not charge-independent unless the coupling constants satisfy the additional restrictions

$$G_1 = -G_2 = G_3/\sqrt{2} \equiv g_1 . \tag{2}$$

As will be seen shortly, the sign of $G_3$ relative to $G_1$ is not observable. It is chosen positive by convention.

Now the only transformations which change the signs of one or more field operators and coupling constants in the Lagrangian (1) and still leave the interaction invariant are [2]

$$\pi^0 \to -\pi^0 , \quad G_1 \to -G_1 , \quad G_2 \to -G_2 \tag{3}$$

$$\pi^+ \to -\pi^+ , \quad G_3 \to -G_3 \tag{4}$$

$$n \to -n , \quad G_3 \to -G_3 \tag{5}$$

$$p \to -p , \quad G_3 \to -G_3 . \tag{6}$$

Let us denote the sign of $G_i$ by $S(G_i)$, the sign of the product $G_iG_j$ by $S(G_iG_j)$, etc. It is clear that none of the signs $S(G_i)$ is an observable, since by transformations (3) and (4) we change the $S(G_i)$ and still leave the Lagrangian invariant. These transformations of the coupling constant signs are only possible because each term in the interaction is linear in the field of one particle. The transformations (5) and (6) add no information about the coupling signs not already contained in (4). It is also clear that other transformations which do more than change the signs of field operators do not add information. For example the transformation

$$G_1 \to -G_2 , \quad G_2 \to -G_1 , \quad p \to n , \quad n \to p , \quad \pi^0 \to -\pi^0 , \quad \pi^\pm \to \pi^\mp \tag{7}$$

[2] If the sign of a field is changed, the sign of the field of the antiparticle must also be changed or the free Lagrangian will not be invariant. Thus, *e.g.*, the transformation $\pi^+ \to -\pi^+$ implies simultaneously $\pi^- \to -\pi^-$.

gives the same information about the observability of the coupling constants signs as does the transformation (3).

Now consider the relative sign $S(G_1G_2)$. This is an observable, since it is not changed by applying either transformation (3) or (4). This is because the total number of times each field appears in the first two terms (taken together) in the Lagrangian is even.

In fact $S(G_1G_2)$ is the only observable sign in the Lagrangian as can be readily seen from transformations (3) and (4). For example, $S(G_2G_3)$ is not an observable because it can be changed by applying either transformation (3) or (4) [3]. By the experimental observation that charge-independence holds to a good approximation in pion-nucleon interactions, we know that the relation (2) must approximately hold. This constitutes an observation that $S(G_1G_2)$ is negative. The result that $S(G_1G_2)$ is the only observable sign is already known, but the reasoning by which we obtained it can be generalized in the following two rules:

*A) The sign of a coupling constant is observable only if the constant multiplies a term in the Lagrangian containing all fields an even number of times.*

This follows because if any field $\varphi$ is contained an odd number of times, the transformation

$$\varphi \to -\varphi\,, \qquad \text{(coupling constant)} \to -\,\text{(coupling constant)}$$

leaves the Lagrangian invariant.

*B) The relative sign of two (or more) coupling constants is observable only if the constants multiply terms such that all fields appear an even number of times when the terms are considered together.*

This is an obvious extension of rule *A*).

In the Lagrangian (1) we saw that the $S(G_i)$ are not observables and that $S(G_1G_2)$ is. Similarly we can see from the rules that $S(G_1G_3)$, $S(G_2G_3)$ and $S(G_1G_2G_3)$ are not observables. Applying the rules to all four terms we see that $S(G_1G_2G_3^2)$ is an observable. However, this is nothing new, since

$$S(G_1G_2G_3^2) = S(G_1G_2)\,S(G_3^2)\,.$$

But $S(G_i^2)$ is always positive, and we already know that $S(G_1G_2)$ is observable. Therefore the observable $S(G_1G_2G_3^2)$ is not independent of $S(G_1G_2)$. It is in this sense that $S(G_1G_2)$ is the only observable sign in the Lagrangian (1).

[3] If the Lagrangian is not given by (1), but instead contains the derivative of the pion field, none of the considerations given above are altered. However, a Lagrangian containing both direct and derivative terms would require more coupling constants and more signs would be observable.

### 3. – Meson-baryon and meson-meson couplings.

We here consider a meson-baryon interaction which is linear in the fields of the $\pi$ and K-mesons and bilinear in the baryon fields. The interaction, which has been considered previously [4], is of the form

$$L = g_1 \mathcal{N}\mathcal{N}\pi + g_2(\bar{\Lambda}\Sigma\pi + \text{H.c.}) + g_3\bar{\Sigma}\Sigma\pi + g_4\bar{\Xi}\Xi\pi + \tag{8}$$

$$+ g_5(\mathcal{N}\Lambda K + \text{H. c.}) + g_6(\mathcal{N}\Sigma K + \text{H. c.}) + g_7(\bar{\Xi}\Lambda\bar{K} - \text{H. c.}) - g_8(\bar{\Xi}\Sigma\bar{K} - \text{H. c.}),$$

where again the symbol for a particle denotes the field which annihilates it, H.c. means Hermitian conjugate, and the $g_i$ $(i = 1, \ldots, 8)$ are coupling constants. We have suppressed the behavior of the interaction in ordinary and isotopic spin spaces; *i.e.*, we have omitted $\gamma$ matrices, possible derivatives in the interaction, etc., as these do not affect our considerations. We have assumed not only that the interaction is invariant under charge conjugation and space and time reflections, but that it is charge-independent as well. If any of these laws do not hold, additional coupling constants must be introduced and additional signs (and perhaps phases as well [5]) become observables.

Applying the rules *A*) and *B*) of the previous section, we obtain the result that there are four independent observable relative signs [6]. We see this as follows:

We use the abbreviation $S(i) = S(g_i)$. Also in writing $S(ij \ldots)$, we assume all $i, j \ldots$ are different. Now none of the $S(i)$, $i = 1, \ldots, 8$ are observables, since all terms in the Lagrangian contain a meson field to an odd power. Similarly there are no observables $S(ijk \ldots)$ to be formed from an odd number of coupling constants. The only observables of the form $S(ij)$ are

$$S(13), \quad S(14), \quad \text{and} \quad S(34). \tag{9}$$

However, only two of these are independent, since any two can be written as the product of the third, *e.g.*

$$S(13)\,S(14) = S(34). \tag{10}$$

---

[4] See, *e.g.*, B. D'Espagnat and J. Prentki: *Nucl. Phys.*, **1**, 33 (1956).

[5] See, *e.g.*, B. D'Espagnat, J. Jauch and Y. Yamaguchi: *Strange Particle Physics*, CERN 59-35 (1959), p. 192.

[6] This result has been obtained previously by explicit examination of the transformations which leave the Lagrangian invariant. D. B. Lichtenberg: *Ann. Phys.* (New York) **12**, 264 (1961).

Turning to products of four coupling constants, the only observables are

$$S(i256)\,, \quad S(i278)\,, \quad S(5678)\,, \qquad i = 1, 3, 4. \tag{11}$$

These are seven observables, but they are not all independent. In fact it is easy to see that they can all be constructed from the independent observables

$$S(13)\,, \quad S(14)\,, \quad S(1256)\,, \quad S(1278)\,. \tag{12}$$

Of the products of six $g$'s, the only five observables are

$$S(123456)\,, \quad S(123478)\,, \quad S(ij5678)\,, \qquad i, j = 1, 3, 4, \tag{13}$$

and these are not independent of the observables given in (12). There is only one sign containing all 8 $g$'s, and it is not an observable. Thus, we have exhausted the possibilities and found only 4 independent observable signs. In particular, the set (12) is complete in the sense that all observable signs in the Lagrangian (8) can be given in terms of this set.

Now consider the set of eight signs $S(i)$. We have seen that they are not themselves observable, and that only 4 independent observables can be constructed from them. This means that four coupling constants can be defined to be positive, provided the choice is consistent with (12). (The signs $S(1)$ and $S(3)$, for example, cannot both be defined positive, since $S(1)S(3) = S(13)$, an observable.) The particular definition used in reference [6] is

$$S(1) = S(2) = S(5) = S(7) = +\,.$$

With this definition, the remaining four signs

$$S(3)\,, \quad S(4)\,, \quad S(6)\,, \quad S(8)$$

are observables.

If any of the coupling constants are zero, the set of independent observables in (12) must be modified. This can be done by putting the appropriate signs in (9), (11) and (13) equal to zero, and picking a new complete independent set.

We note that a choice for the signs of the coupling constants can always be found so that at most two are negative. For example suppose $S(1)$ is defined positive. Then if $S(3)$ and $S(4)$ turn out to be negative, $S(3)$ can be redefined to be positive instead of $S(1)$. With the new definition, only $S(1)$ (of $S(1)$, $S(2)$, and $S(3)$) will be negative. Similarly, suppose $S(2)$ is defined positive; then if $S(6)$ and $S(8)$ turn out negative, $S(6)$ can be redefined to be

positive, and then only $S(2)$ (of $S(2)$, $S(6)$ and $S(8)$) will be negative. However, these choices can be made only after the observable relative signs are determined *from experiment*. We cannot make an *a priori* choice which will insure that at most 2 signs are negative: as many as four may be.

We now consider additional interaction terms. Suppose there are meson-meson interactions of the form

$$L = f_1\pi\pi\pi\pi + f_2\pi\pi\bar{K}K + f_3\bar{K}\bar{K}KK + f_4\bar{K}K^*\pi\,, \tag{14}$$

where $K^*$ is a possible excited state of the K-meson with parity opposite to the ground state ([7]). Then, from the rules, $S(f_1)$, $S(f_2)$, and $S(f_3)$ are observables, but $S(f_4)$ is not. Clearly the presence of such interactions does not change the arguments about the meson-baryon coupling constants considered previously. However, if there exist further meson-baryon interaction terms, perhaps involving other baryons such as the Y* ([8]), more relative signs may become observable. We need only write down the new interaction terms and apply the rules to obtain the new observable signs.

We conclude this section by noting that the weak interactions may provide additional information about the signs of strong interaction coupling constants ([9]). For example, consider an interaction of the form

$$L = g_1\bar{\mathcal{N}}\mathcal{N}\pi + g_2\bar{\Lambda}\Sigma\pi + f_5\bar{\Lambda}\mathcal{N}\pi + f_6\bar{\Sigma}\mathcal{N}\pi\,,$$

where $f_5$ and $f_6$ are weak interaction coupling constants ([10]). Then it follows that the relative sign $S(12f_5f_6)$ is an observable. But $f_5$ and $f_6$ may both be defined to be positive, since $S(f_5)$, $S(f_6)$, and $S(f_5f_6)$ are not observables. If this is done, the sign $S(12)$ becomes an observable. Thus Feldman, Matthews and Salam ([9]) discuss the determination of $S(12)$ by consideration of decay processes, assuming that $S(f_5f_6)$ is positive at the outset.

---

([7]) Evidence for this state (parity undetermined) was reported at the Conference on Strong Interactions (Berkeley, Dec. 1960).

([8]) Experimental information about the Y* was reported at the Conference on Strong Interactions (Berkeley, Dec. 1960).

([9]) See G. Feldman, P. Matthews and A. Salam: *Phys. Rev.*, **121**, 302 (1961) and references contained therein. Also A. Pais: *Nuovo Cimento*, **18**, 1003 (1960).

([10]) The weak interaction terms in the above Lagrangian should be considered as phenomenological, with $f_1$ and $f_2$ being effective coupling constants. For example the interaction $\Lambda \to \mathcal{N} + \pi$ may go as a two step process via the interactions $\bar{\Lambda}\mathcal{N}\bar{\mathcal{N}}\mathcal{N}$, $\bar{\mathcal{N}}\mathcal{N}\pi$, or as a three step process via the interactions $\bar{\Lambda}\mathcal{N}W$, $\bar{\mathcal{N}}\mathcal{N}W$, $\bar{\mathcal{N}}\mathcal{N}\pi$, where W is an intermediate vector boson. But these possibilities do not affect our considerations.

## 4. – Validity of assumptions and possible experimental tests.

The previous results have been based on the Lagrangian formalism in local field theory, a formalism which is unsatisfactory in several respects. However, the same conclusions follows in the dispersion relation approach to particle physics. For example, the coupling constant appearing in the Lagrangian describing the interaction of a pion and nucleon also appears explicitly in a dispersion relation for pion-nucleon scattering. In the particular case of pion-nucleon scattering, only the square of the coupling constant appears in the dispersion relation (assuming charge independence). However, if the sign of any coupling constant is an observable, it should be possible to obtain a dispersion relation in which the coupling constant appears linearly. Similarly, it should be possible to obtain dispersion relations which depend on every observable relative sign.

Alternatively, the constants may be regarded simply as parameters appearing in a phenomenological theory. This may be true in particular for the coupling constants appearing in the Lagrangian (14). One might have a hope of ultimately deriving all meson-meson interactions from a meson-baryon interaction such as that given in (8), but at present this cannot be done. Therefore the coupling constants $f_i$ $(i = 1, ..., 4)$ may be regarded as additional parameters to be determined from experiment.

It may be possible (but not easy) to measure the relative sign $S(13)$ with existing experimental techniques. The most direct experiment is to measure $\Sigma^+$-p elastic scattering at energies above 10 MeV ([11]).

The Lagrangian (8) leads to a potential between the $\Sigma$ and nucleon which, at large separations of the particles, is of Yukawa form. The sign of this potential depends on $S(13)$.

First assume $S(13)$ is positive. Then the Yukawa potential between the $\Sigma^+$ and proton is attractive in the singlet state and repulsive in the triplet state (both central and tensor parts). If $S(13)$ is negative, the Yukawa potential simply reverses sign.

Unfortunately the sign of the potential can be calculated only at large separations of the two baryons. At small separations the potential depends in an unknown way on $S(13)$. But scattering in states of zero orbital angular momentum depends on the behavior of the potential at small distances. Therefore, it is necessary to measure scattering in states with orbital angular momentum $l \geqslant 1$, to obtain information about $S(13)$. The simplest procedure in

---

([11]) A more detailed discussion of experiments to measure S(13) has been given by M. T. VAUGHN: *Nuovo Cimento*, **18**, 178 (1960).

principle is to determine the sign of the singlet $p$-, $d$-, or $f$-wave phase shift at moderate energy. (The higher the $l$ of the phase shift, the less sensitive it is to the form of the interaction at small distances.) The sign of these phase shifts is the same as the sign of $S(13)$ provided there is no bound state with the same $l$. In view of the well known difficulties in obtaining the signs of phase shifts, it will probably be necessary to make Coulomb interference and polarization measurements. The singlet phase shift should be measured, since in the triplet state the tensor potential complicates the relationship between potential and phase shifts.

A measurement of $S(14)$ is even more difficult. The sign of the potential between either the $\Xi^0$-p or $\Xi^-$-n depends on the sign of $S(14)$ in the same way that the $\Sigma^+$-p potential depends on $S(13)$. But as a matter of practice, experiments with either a neutral $\Xi$ as incident particle or a deuteron as target particle will be hard to interpret. The $\Xi^-$-p system is a mixture of isotopic spin states ($I = 0, 1$) so that twice as many phase shifts are necessary to specify the scattering as for $\Sigma^+$-p. In the $I = 1$ state the relation between the sign of the phase shift and the observable relative coupling constant sign is the same as for $\Sigma^+$-p. In the $I = 0$ state, the relation between potential and the coupling constant sign is reversed.

The determination of the two observable signs $S(1256)$ and $S(1278)$ by means of strong interaction experiments presents theoretical as well as experimental difficulties. In the first place the relative parities of the K, $\Lambda$, and $\Sigma$ must be determined. Furthermore, since K-meson coupling constants are involved, it seems that relevant experiments would have to probe distances of the order of the K-meson Compton wavelength. The predictions of the theory are at present unreliable at such small distances.

The signs of coupling constants such as $f_1$ appearing in the pion-pion interaction are also observable. Chew, Mandelstam and Noyes [12] have approximately evaluated double dispersion relations for this case, and obtained the result that a pion-pion interaction depends on the sign as well as the magnitude of the coupling constant. However, the theory is not yet good enough to make possible a clear cut interpretation of experiment.

* * *

The author is grateful to Professors K. M. Case, I. Goldberg, J. S. Kovacs and H. McManus for helpful discussions.

[12] G. Chew, S. Mandelstam and H. Noyes: *Phys. Rev.*, **119**, 478 (196 ).

***Note added in proof.***

After this work was completed, the author came upon a paper by I. Yu. Kobzarev and L. B. Okun: *Žurn. Ėks. Teor. Fiz.*, **39**, 210 (1960). [Translation: *Soviet Physics JEPT*, **12**, 150 (1961)] containing essentially the same results as given in rules *A* and *B*. Kobzarev and Okun conclude that in the Lagrangian of eq. (8), only four relative coupling constant signs are observable, in agreement with the conclusions given here and in ref. (6).


## RIASSUNTO (*)

Esponiamo i criteri di osservabilità dei segni delle costanti di accoppiamento che appaiono nelle interazioni delle particelle elementari. I risultati vengono applicati a particolari lagrangiani delle interazioni forti. Mostriamo che, nella interazione mesone-barione, comunemente considerata, con otto costanti di accoppiamento, solo quattro segni relativi (indipendenti) sono osservabili. Discutiamo brevemente i segni delle costanti di accoppiamento in alcune interazioni mesone-mesone. Facciamo menzione di alcune difficoltà nella misura dei segni osservabili.


(*) *Traduzione a cura della Redazione.*

# Resonance Poles and the Reaction Matrix (*).

R. Oehme

*The Enrico Fermi Institute for Nuclear Studies - Chicago, Ill.*
*Department of Physics, University of Chicago - Chicago, Ill.*

(ricevuto il 6 Febbraio 1961)

**Summary.** — Within the framework of relativistic dispersion theory the reaction matrix ($K$-matrix) is considered as an analytic matrix-function. For the case of two coupled two-particle channels, it is shown that the elements of this matrix are regular functions except for cuts due to dynamical branch lines of the amplitudes and possible isolated poles. Expressing the reaction amplitudes in terms of the elements of the reaction matrix makes it possible to exhibit the Riemann surface of these amplitudes. The connection between resonances and poles of the amplitudes in secondary Riemann sheets is discussed with the help of the reaction-matrix formalism. The possible connection between a resonance and a bound state is considered.

## 1. – Reaction amplitudes.

In this note we shall discuss the connections between a general dispersion theoretic description of reactions with several two-particle channels ([1,2]) and a treatment of the same problem using reaction matrices ([3]). We restrict ourselves to $s$-waves, and we consider the elements of a reaction matrix as analytic functions of the invariant energy variable. These functions coincide with the elements of the conventional $K$-matrix ([4]) in an interval on the real

(*) Work supported in part by the U.S. Atomic Energy Commission.

(1) R. Oehme: *Zeits. f. Phys.* (to be published) and EFINS 5-61. This paper contains further references.

(2) A formulation of the many-channel problem using the $ND^{-1}$ method has been given by J. D. Bjorken: *Phys. Rev. Lett.*, **4**, 473 (1960); M. Nauenberg: *Ph. D. dissertation* (Cornell University, 1960); J. D. Bjorken and M. Nauenberg: *Phys. Rev.* (to be published); R. Blankenbecler: preprint (1960).

(3) R. H. Dalitz and S. F. Tuan: *Ann. Phys.*, **3**, 307 (1960); *Phys. Rev. Lett.*, **2**, 425 (1959).

(4) B. Lippmann and J. S. Schwinger: *Phys. Rev.*, **79**, 469 (1950); M. L. Goldberger: *Phys. Rev.*, **84**, 928 (1951), this paper contains further references.

axis between two consecutive thresholds. To every such interval corresponds a different reaction matrix, the elements of which are real analytic functions. At least for the case of two channels we find that these functions are one-valued at the thresholds of all channels which are open in the characteristic interval, except for possible dynamical branch lines from the left-hand side (*). We can express the elements of the scattering matrix in terms of those of a given reaction matrix, and we obtain in this way an expression in which the Riemann surface of the amplitudes related to the open channels is exhibited. It is mainly because of these properties that the reaction matrix is useful for the discussion of certain types of resonances and threshold properties ([3,5]). On the other hand, the different dynamical singularities of the various channel amplitudes are usually all present in a given element of the reaction matrix ([3]). There can also be isolated poles, but these are not the same as the poles of the amplitudes on secondary Riemann sheets.

We restrict ourselves here to two-channel reactions in order to bring out the analytic properties. With the use of a matrix notation, the generalization of the formalism is straightforward. As in ref. ([1]), we use as a model the reactions

$$\pi + \mathrm{Y} \to \pi + \mathrm{Y}\,, \tag{1.1}$$

$$\overline{\mathrm{K}} + \mathcal{N} \to \overline{\mathrm{K}} + \mathcal{N}\,, \tag{1.2}$$

$$\pi + \mathrm{Y} \to \overline{\mathrm{K}} + \mathcal{N}\,, \tag{1.3}$$

but we ignore all complications connected with spin and isotopics pin. The restrictions resulting from the conservation of baryon number, strangeness, parity, etc., are taken into account. We denote the covariant partial wave amplitudes corresponding to reactions (1.1)–(1.3) by $F$, $K$ and $G$ respectively, and we restrict ourselves to $s$-waves. The scattering amplitudes $F$ and $K$ are related to the physical quantities in the c.m. system by the relations

$$F(s+i0) = \varrho_\pi^{-1}(s+i0)\sin\delta_\pi(s)\exp\left[i\delta_\pi(s)\right], \tag{1.4}$$

$$K(s+i0) = \varrho_{\mathrm{K}}^{-1}(s+i0)\sin\delta_{\mathrm{K}}(s)\exp\left[i\delta_{\mathrm{K}}(s)\right], \tag{1.5}$$

where

$$\varrho_\pi(z) = \frac{2q_\pi(z)}{\sqrt{z}} = \frac{1}{z}\left\{[z-(m_\pi+m_{\mathrm{Y}})^2][z-(m_\pi-m_{\mathrm{Y}})^2]\right\}^{\frac{1}{2}}, \tag{1.6}$$

$$\varrho_{\mathrm{K}}(z) = \frac{2q_{\mathrm{K}}(z)}{\sqrt{z}} = \frac{1}{z}\left\{[z-(m_{\mathrm{K}}+m_{\mathcal{N}})^2][z-(m_{\mathrm{K}}-m_{\mathcal{N}})^2]\right\}^{\frac{1}{2}}. \tag{1.7}$$

---

([5]) P. T. Mathews and A. Salam: *Nuovo Cimento*, **13**, 382 (1959).

(*) The case of $n$ channels as well as other generalizations, will be discussed in a forthcoming paper by C. Iddings.

The roots in eq. (1.6)–(1.7) are defined such that

$$\varrho_\pi^*(z^*) = -\varrho_\pi(z)\,, \qquad \varrho_K^*(z^*) = -\varrho_K(z)\,, \tag{1.8}$$

and that we have $\mathrm{Im}\,\varrho_\pi(z) \geqslant 0$, $\mathrm{Im}\,\varrho_K(z) \geqslant 0$ in the cut $z$-plane.

Within the framework of our model the analytic properties of the functions $F(z)$, $K(z)$, and $G(z)$ have been discussed in ref. [1] [6]. We are mainly interested in the branch lines $s \geqslant (m_\pi + m_Y)^2$ and $s \geqslant (m_K - m_N)^2$, and we neglect the singularities associated with the intermediate state $Y\pi\pi$ and higher many-particle states. All dynamical singularities are to the left of the point $s = (m_\pi + m_Y)^2$ except for the cut in the $z$-plane of the function $K(z)$ which is due to the reaction $K + \bar{K} \to \mathcal{N}' + \bar{\mathcal{N}}'$. An intermediate state of the reaction $K + \bar{K} \to \mathcal{N}' + \bar{\mathcal{N}}'$ with energy $t \geqslant 4\mu^2$ gives rise to a dynamical branch line (left-hand cut) of $K(z)$ which terminates at

$$S_L(t) = \left\{(m_{\mathcal{N}}^2 - \tfrac{1}{4}t)^{\frac{1}{2}} + (m_K^2 - \tfrac{1}{4}t)^{\frac{1}{2}}\right\}^2. \tag{1.9}$$

For $t = 4m_\pi^2$, we find

$$S_L(4m_\pi^2) \equiv S_L(K) > (m_\pi + m_Y)^2\,; \tag{1.10}$$

the left-hand branch line overlaps considerably with the pion hyperon cut, but $S_L(K) < (m_K + m_N)^2$.

The continuation of the amplitudes $F$, $K$ and $G$ through the right-hand branch lines can be achieved with the help of the unitarity requirements. We find that these functions are analytic on a Riemann surface with four sheets. We denote the physical sheet by I; the sheets II and III are connected with I along the real axis in the intervals $(m_\pi + m_Y)^2 < s < (m_K + m_N)^2$ (or $S_L(K) < s < (m_K + m_N)^2$ for $K$) and $s > (m_K + m_N)^2$ respectively. In the same intervals sheet IV is connected with the sheets III and II respectively, (see Fig. 1 of ref. [1]). The physical branches of the functions $F$, $K$ and $G$ have a pole at $z = m_Y^2$ due to a one-hyperon intermediate state. In sheets II, III and IV isolated poles of the amplitudes occur only for $1 + 2i\varrho_\pi F = 0$ in II, for $(1 + 2i\varrho_\pi F)\cdot(1 + 2i\varrho_K K) + 4\varrho_\pi\varrho_K G^2 = 0$ in III and for $1 + 2i\varrho_K K = 0$ in IV. Generally all branches of the amplitudes, except $F^{\mathrm{I}}$, $G^{\mathrm{I}}$, $F^{\mathrm{III}}$ and $G^{\mathrm{III}}$, have a dynamic branch line ending at $z = S_L(K)$.

For later reference we reproduce here the eq. (3.1)–(3.3) and (3.10)–(3.12) of ref. [1] for the continuation of the amplitudes into sheets II and III. They are

$$F^{\mathrm{II}} = \frac{F}{1 + 2i\varrho_\pi F}\,, \quad K^{\mathrm{II}} = K - 2i\varrho_\pi \frac{G^2}{1 + 2i\varrho_\pi F}\,, \quad G^{\mathrm{II}} = \frac{G}{1 + 2i\varrho_\pi F}\,, \tag{1.11}$$

[6] See also M. Nauenberg: ref. [2]; S. MacDowell: *Phys. Rev.*, **116**, 774 (1959).

and

$$(1.12)\qquad \begin{cases} F^{\mathrm{III}} = \dfrac{F(1+2i\varrho_{\mathrm{K}}K) - 2i\varrho_{\mathrm{K}}G^2}{(1+2i\varrho_\pi F)(1+2i\varrho_{\mathrm{K}}K) + 4\varrho_\pi\varrho_{\mathrm{K}}G^2}\,, \\ K^{\mathrm{III}} = \dfrac{K(1+2i\varrho_\pi F) - 2i\varrho_\pi G^2}{(1+2i\varrho_\pi F)(1+2i\varrho_{\mathrm{K}}K) + 4\varrho_\pi\varrho_{\mathrm{K}}G^2}\,, \\ G^{\mathrm{III}} = \dfrac{G}{(1+2i\varrho_\pi F)(1+2i\varrho_{\mathrm{K}}K) + 4\varrho_\pi\varrho_{\mathrm{K}}G^2}\,. \end{cases}$$

## 2. – The reaction matrix.

For the definition of the reaction matrix, it is convenient to introduce a $T$-matrix

$$(2.1)\qquad \boldsymbol{T} = \begin{pmatrix} F & G \\ G & K \end{pmatrix},$$

and a diagonal matrix of the form

$$(2.2)\qquad \boldsymbol{\rho} = \begin{pmatrix} \varrho_\pi & 0 \\ 0 & \varrho_{\mathrm{K}} \end{pmatrix}.$$

Then the unitarity relations for $s \geqslant (m_{\mathrm{K}} + m_{\mathrm{Y}})^2$, $z = s + i0$ can be written in the familiar form

$$(2.3)\qquad \boldsymbol{T}\boldsymbol{\rho}\boldsymbol{T}^\dagger = \frac{1}{2i}\,(\boldsymbol{T} - \boldsymbol{T}^\dagger)\,.$$

We introduce a reaction matrix $\boldsymbol{R}$ by [3,4]

$$(2.4)\qquad \boldsymbol{T} - \boldsymbol{R} = i\boldsymbol{R}\boldsymbol{\rho}\boldsymbol{T}\,,$$

and, writing

$$(2.5)\qquad \boldsymbol{R} = \begin{pmatrix} \varphi & \gamma \\ \gamma & \varkappa \end{pmatrix},$$

we may solve eq. (2.4) and express the elements of $\boldsymbol{R}$ by the covariant amplitudes $F$, $K$ and $G$. The result is

$$(2.6)\qquad \varphi = \frac{F(1+i\varrho_{\mathrm{K}}K) - i\varrho_{\mathrm{K}}G^2}{(1+i\varrho_\pi F)(1+i\varrho_{\mathrm{K}}K) + \varrho_\pi\varrho_{\mathrm{K}}G^2}\,,$$

$$(2.7)\qquad \varkappa = \frac{K(1+i\varrho_\pi F) - i\varrho_\pi G^2}{(1+i\varrho_\pi F)(1+i\varrho_{\mathrm{K}}K) + \varrho_\pi\varrho_{\mathrm{K}}G^2}\,,$$

$$(2.8)\qquad \gamma = \frac{G}{(1+i\varrho_\pi F)(1+i\varrho_{\mathrm{K}}K) + \varrho_\pi\varrho_{\mathrm{K}}G^2}\,.$$

So far the elements $q$, $x$ and $\gamma$ have been defined only for $z=s-i0$, $s\geqslant(m_K-m_N)^2$, but we may use the known analytic properties of the amplitudes and the functions $\varrho_\pi(z)$ and $\varrho_K(z)$ in order to continue these elements into the complex $z$-plane. We find that $q(z)$, $x(z)$, and $\gamma(z)$ are real analytic functions. Because of $\boldsymbol{R}^T=\boldsymbol{R}$ we write this:

$$\boldsymbol{R}^\dagger(z^*)=\boldsymbol{R}(z)\,. \tag{2.9}$$

From the definition (2.4) of the reaction matrix and the unitarity condition (2.3), it follows that $\boldsymbol{R}(s+i0)$ is hermitian for $s\geqslant(m_K+m_N)^2$. If we combine this with the reality condition (2.9), we find that

$$\boldsymbol{R}(s+i0)-\boldsymbol{R}(s-i0)=0$$

in the physical region of all three reactions. Hence there is no branch line for $s\geqslant(m_K+m_N)^2$ and we have

$$\boldsymbol{R}^{III}(z)=\boldsymbol{R}(z)\,. \tag{2.10}$$

Of course, the relations (2.10) may also be checked by inserting eq. (1.12) for $F^{III}$, $K^{III}$ and $G^{III}$ into the relations

$$\varphi^{III}=\frac{F^{III}(1-i\varrho_K K^{III})+i\varrho_K G^{III^2}}{(1-i\varrho_\pi F^{III})(1-i\varrho_K K^{III})+\varrho_\pi\varrho_K G^{III^2}}\,, \tag{2.11}$$

etc.

Note that it follows from eq. (1.16) and (1.17) that $\boldsymbol{\rho}^{III}=-\boldsymbol{\rho}$, or

$$\varrho_\pi^{III}=-\varrho_\pi\,,\qquad \varrho_K^{III}=-\varrho_K\,. \tag{2.12}$$

Corresponding relations hold for the other sheets:

$$\varrho_\pi^{II}=-\varrho_\pi\,,\qquad \varrho_K^{II}=\varrho_K\,, \tag{2.13}$$

$$\varrho_\pi^{IV}=\varrho_\pi\,,\qquad \varrho_K^{IV}=-\varrho_K\,. \tag{2.14}$$

We can also compute $q^{II}(z)$, $x^{II}(z)$ and $\gamma^{II}(z)$ using eq. (2.6)–(2.8), (2.13) and the expressions (1.11) for $F^{II}$, etc. Again we find that

$$\boldsymbol{R}^{II}(z)=\boldsymbol{R}(z)\,. \tag{2.15}$$

The elements of the reaction matrix $\boldsymbol{R}$ are regular and one valued along the real axis except for the dynamical singularities in the region $s\leqslant S_L(K)$ and possible poles due to zeros of the denominator in eq. (2.6)–(2.7). Ignoring

subtractions we write

$$\varphi(z) = \frac{1}{\pi}\int_{-\infty}^{S_L(K)} \mathrm{d}z' \, \frac{\alpha_\varphi(z')}{z'-z} + \text{possible pole terms}. \tag{2.16}$$

Here the integration path runs along the dynamical branch lines of all three amplitudes. The weight function $\alpha_\varphi(z')$ can be computed from (2.6)–(2.8) and the corresponding weight functions of the amplitudes $F$, $K$ and $G$. Of special interest is the interval $(m_Y+m_\pi)^2 < s \leqslant S_L(K)$ on the real axis where we have the overlapping dynamical branch line of $K(z)$. If we denote the weight of this branch line by $A_K(s)$ such that

$$\operatorname{Im} K(s+i0) = \varrho_\pi G(s+i0)\, G^*(s+i0) + A_K(s)\,, \tag{2.17}$$

for $(m_\pi+m_Y)^2 < s \leqslant S_L(K)$, then we find for $\alpha_\varphi(s)$ in the same interval:

$$\alpha_\varphi(s) = \operatorname{Im}\varphi(s+i0) = -\frac{\varrho_\pi^2(s)\gamma^2(s+i0)A_K(s)}{1+2\varrho_K(s)A_K(s)(1-i\varrho_K(s)\varkappa(s+i0))}\,. \tag{2.18}$$

The functions $\varkappa(z)$ and $\gamma(z)$ have representations analogous to eq. (2.16). For the weight functions in the interval $(m_\pi+m_Y)^2 < s \leqslant S_L(K)$, we find

$$\alpha_\varkappa(s) = \operatorname{Im}\varkappa(s+i0) = \frac{(1-i\varrho_K(s)\varkappa(s+i0))^2 A_K(s)}{1+2\varrho_K(s)A_K(s)(1-i\varrho_K(s)\varkappa(s+i0))}\,, \tag{2.19}$$

and

$$\alpha_\gamma(s) = \operatorname{Im}\gamma(s+i0) = \frac{2\varrho_K(s)\gamma(s+i0)(1-i\varrho_K(s)\varkappa(s+i0))A_K(s)}{1+2\varrho_K(s)A_K(s)(1-i\varrho_K(s)\varkappa(s+i0))}\,. \tag{2.20}$$

We note that all elements of the reaction matrix $\boldsymbol{R}$ have branch lines for $s \leqslant S_L(K)$ [3], but for $q(z)$ the weight in the region $S_L(F) < s \leqslant S_L(K)$ is proportional to $\gamma^2$ and vanishes in the limit of zero coupling between the channels; in our model the limit $S_L(F)$ is given by $S_L(F) = m_Y^2 + 2m_\pi^2$. If we assume that the « forces » are known, which, in our formalism, corresponds to the knowledge of the weight functions $A_K$, $A_G$ and $A_F$ along the dynamical branch lines, then we could in principle use the dispersion representations corresponding to eq. (2.16) as integral equations for the elements of the reaction matrix. From a practical point of view one may hope to formulate approximate equations based upon some simple model for the forces.

It has been pointed out by Dalitz and Tuan [3] that the reaction matrix is especially useful if its elements are slowly varying functions in the neigh-

borhood of a physical threshold corresponding to $s=(m_K+m_N)^2$ in our model. There the functions $q$, $\varkappa$ and $\gamma$ can be expanded into power series, and if we exclude artificial pole terms in eq. (2.16) and related representations, we see that the radius of convergence of such an expansion around $z=(m_K+m_N)^2$ would be given by $(m_K+m_N)^2-S_L(K)$. This is familiar if we recall that in the language of potentials the lenght $S_L^{-\frac{1}{2}}(K)$ is a measure of the range of the forces. The extent of the region where the functions $q(z)$, etc., can be approximated by their threshold values depends, of course, very much upon the weight functions $\alpha_q(s)$, etc., which in turn are proportional to $A_K(s)$.

The reaction matrix, as we have introduced it in eq. (2.4), is defined as an analytic matrix-function which coincides with the conventional $K$-matrix in the interval $s>(m_K+m_N)^2$ (we have omitted the branch lines corresponding to higher mass channels). Because we have only two channels, the corresponding « matrix », which is characteristic for the interval $(m_\pi+m_Y)^2\leqslant s<(m_K+m_N)^2$, is simply given by the function [7]

$$\varphi_1(z)=\frac{F(z)}{1+i\varrho_\pi(z)F(z)}. \tag{2.21}$$

It may be expressed [3] in terms of the elements of $\boldsymbol{R}(z)$:

$$\varphi_1(z)=q(z)+i\varrho_K(z)\frac{\gamma^2(z)}{1-i\varrho_K(z)\varkappa(z)}. \tag{2.22}$$

We see from eq. (2.21) and (2.22) that $\varphi_1(z)$ is real and one valued in the interval $(m_\pi+m_Y)^2<s<(m_K+m_N)^2$ of the real axis, where it coincides with the expression

$$\varphi_1(s)=\varrho_\pi^{-1}(s)\operatorname{tg}\delta_\pi(s).$$

In addition to the $\bar{K}N$-cut and the dynamical branch lines for $s\leqslant S_L(F)$, the function $\varphi_1(z)$ can have isolated poles due to zeros of the denominator $1+i\varrho_\pi F$. For reasons of completeness we mention only that the function $\varrho_K^{-1}\operatorname{tg}\delta_K=K(1+i\varrho_K K)^{-1}$ has a branch point at $s=(m_\pi+m_Y)^2$ and none at $s=(m_K+m_N)^2$.

## 3. – Resonance poles.

Let us now express the elements of the $T$-matrix in terms of those of the $R$-matrix. In the region $z=s+i0$, $s>(m_K+m_N)^2$, where all thea mplitudes

(7) K. SYMANZIK: *Journ. Math. Phys.*, **1**, 249 (1960); W. ZIMMERMANN: (to be published); R. OEHME: *Phys. Rev.* (March 15, 1961).

ISTITUTO DI FISICA TEORICA - UNIVERSITA' DI NAPOLI

# INTERNATIONAL SCHOOL OF PHYSICS

Under the Auspices of the Italian Physical Society and of the University of Naples

# COURSE ON BIOPHYSICS

*July 10 to 23, 1961 - Naples (Italy)*

This Course, which is part of the Advanced Study institute programme of NATO, will be held in Naples in the period from July 10th to 23rd, 1961. It is dedicated to MOLECULAR BIOLOGY, its main scope is that of bringing together leading workers in the field and advanced students, with a view to stimulating informal discussions among all participants; a limited number of lectures and seminars will serve to bring into focus the arguments of interest.

Invitations have been extended to a large number of experts; a list of lecturers will be made known as soon as possible, together with a tentative program. The detailed program will be decided in a general discussion to be held on July 10th.

All lectures, seminars, conferences and general discussions will be held in English or French.

Those wishing to attend the School should send, not later than May 31st, 1961 an application to the Director of the School giving the following information: 1) Christian name and surname; 2) Date and place of birth; 3) Present address; 4) Degrees and other qualifications obtained, with name of University in each case; 5) Present professional activity; 6) List of publications; 7) Standard of knowledge of English and French — written and spoken; 8) Whether intending to stay in Naples unaccompained or with members of family and, in the former case, whether willing to share a room with other students. The applicant should send, together with his application, a note of reference from a University professor testifying the applicant's interest and preparation for the activities of the School.

Each application will be considered by the Director of the School on the basis of the information submitted, with regard also to a fair distribution of the places available among students of various nations. The decisions on the admittance to the School will be made known to the applicants by June 10th, 1961; in a few exceptional cases this date may be anticipated.

Students are requested to register at the Istituto di Fisica Teorica, Mostra d'Oltremare, Padiglione 19, in the afternoon of July 9th.

The total fee to be paid by students to the management of the School for attendance, hotel accommodation and full board is L. 40.000 if the student has single room accommodation and L. 35.000 if the student shares room accommodation with other students or with members of his family.

Fees should be paid not later than 5 days after the beginning of the course to the management of the School in Italian currency. A limited number of scholarships may be granted to students whose economic conditions might otherwise prevent them from attending the School.

The School will do everything possible to find suitable accomodation in local hotels for members of families accompanying the students. Members of student's families may avail themselves of the catering arrangements being organized for the School. All expenditures involved in the hotel accommodation, board etc. for students relatives will be payable separately. It is estimated that L. 3.500 per person per day may suffice.

For further information apply to the Director of the School, Professor Eduardo R. Caianiello, Istituto di Fisica Teorica, Mostra d'Oltremare, Pad. 19, Napoli.

The Director of the School
E. R. CAIANIELLO

are physical, we obtain the familiar relations

$$F = \frac{\varphi(1 - i\varrho_K\varkappa) + i\varrho_K\gamma^2}{(1 - i\varrho_\pi\varphi)(1 - i\varrho_K\varkappa) + \varrho_\pi\varrho_K\gamma^2}, \tag{3.1}$$

$$K = \frac{\varkappa(1 - i\varrho_\pi\varphi) + i\varrho_\pi\gamma^2}{(1 - i\varrho_\pi\varphi)(1 - i\varrho_K\varkappa) + \varrho_\pi\varrho_K\gamma^2}, \tag{3.2}$$

$$G = \frac{\gamma}{(1 - i\varrho_\pi\varphi)(1 - i\varrho_K\varkappa) + \varrho_\pi\varrho_K\gamma^2}. \tag{3.3}$$

In Section **2** we have discussed the analytic properties of the reaction matrix. We can now use these properties in order to continue the expressions (3.1)–(3.3) onto the complete Riemann surface with four sheets, which is associated with the branch points $z = (m_\pi + m_Y)^2$ and $z = (m_K + m_N)^2$. In sheet I, the physical sheet, the function $F(z)$, $K(z)$ and $G(z)$ are given by eq. (3.1)–(3.3) in terms of the elements $\varphi(z)$, $\varkappa(z)$, $\gamma(z)$ and the functions $\varrho_\pi(z)$, $\varrho_K(z)$ as defined in eq. (1.6)–(1.8). The cuts related to the branch points mentioned above are completely exhibited in eq. (3.1)–(3.3) by the right-hand branch lines of the square roots contained in $\varrho_\pi$ and $\varrho_K$. Hence the continuation into the other sheets is straightforward on the basis of eq. (2.12)–(2.14); so we have as an example

$$K^{II} = \frac{\varkappa(1 + i\varrho_\pi\varphi) - i\varrho_\pi\gamma^2}{(1 + i\varrho_\pi\varphi)(1 - i\varrho_K\varkappa) - \varrho_\pi\varrho_K\gamma^2}. \tag{3.4}$$

We know that in our model the amplitudes have no isolated poles in the physical sheet except the one at $z = m_Y^2$ for $s$-wave amplitudes. Hence the functions $\varphi$, $\varkappa$ and $\gamma$ must be such that $(1 - i\varrho_\pi q)(1 - i\varrho_K\varkappa) + \varrho_\pi\varrho_K\gamma^2 \neq 0$ except at the point $z = m_Y^2$, where the hyperon pole appears in the $s$-wave amplitudes as a zero of the denominator in eq. (3.1)–(3.3). In the framework of our model it appears to be very artificial to describe the hyperon pole $s = m_Y^2$ in such an implicit way whereas the related branch point $s = (m_Y + m_\pi)^2$ has been exhibited explicitly. But the reaction matrix formalism is mainly of interest for the parametrization of the amplitudes in the neighborhood of the $\bar{K}N$-threshold, and therefore we shall not try to modify it because of the hyperon pole.

Let us now consider resonance poles on the second Riemann sheet. These are pairs of conjugate poles which are located sufficiently near the interval $(m_\pi + m_Y)^2 < s < (m_K + m_N)^2$ of the real axis to produce a sharp maximum in the absorption parts of the amplitudes. From the continuation of eq. (3.1)–(3.3) into sheet II we see that these poles must be zeros of the expression

$$(1 + i\varrho_\pi\varphi)(1 - i\varrho_K\varkappa) - \varrho_\pi\varrho_K\gamma^2, \tag{3.5}$$

which corresponds to zeros of $1+2i\varrho_\pi F$. It is of interest to consider the possible movement of a pair of resonance poles at $z=z_0$, $z_0^*$ in the limit of vanishing coupling between the channels. We assume that we can characterize this limit by $\gamma(z)\to 0$ for all $z$. Of course, the functions $\varphi$ and $\varkappa$ will also change and so does the position of the poles $(z_0, z_0^*)$. We denote the limiting functions by $\varphi_0(z)$ and $\varkappa_0(z)$. From eq. (3.1)–(3.3) and their continuations into sheet II we find then for $\gamma\to 0$

$$F\to\frac{\varphi_0}{1-i\varrho_\pi\varphi_0}\,,\qquad F^{\mathrm{II}}\to\frac{\varphi_0}{1+i\varrho_\pi\varphi_0}\,, \tag{3.6}$$

and

$$K^{\mathrm{II}}\to K\to\frac{\varkappa_0}{1-i\varrho_{\mathrm{K}}\varkappa_0}\,. \tag{3.7}$$

Let us assume that the physical situation $(\gamma\neq 0)$ is such that $\gamma$ is small in comparison with $\varphi$ and $\varkappa$, especially in the neighborhood of the zeros of eq. (3.5) at $z=z_0$, $z_0^*$ corresponding to the resonance poles. Then these zeros must be due to $(1+i\varrho_\pi\varphi)\sim 0$ or to $(1-i\varrho_{\mathrm{K}}\varkappa)\sim 0$. Assuming that the limit $\gamma\to 0$ does not change very much the initial position of the singularities, we find that the poles either go over into a pair of complex resonance poles of the decoupled $\pi\mathrm{Y}$ system, or the conjugate poles collapse into a real single particle pole of the uncoupled $\bar{\mathrm{K}}\mathcal{N}$' system. In the first case the poles are due to zeros of the expression $1+i\varrho_\pi\varphi_0$. Because $\varphi_0$ is real on the real axis, we cannot have a real zero of $1+i\varrho_\pi\varphi_0$ in the region of interest. This is a consequence of the unitarity condition in the $\pi\mathrm{Y}$ channel. In the second case the single particle pole is described by a zero of $1-i\varrho_{\mathrm{K}}\varkappa_0$. The expression $1-i\varrho_{\mathrm{K}}\varkappa_0$ can vanish only on the real axis because the $\bar{\mathrm{K}}\mathcal{N}$'-amplitude $K(z)$ cannot have complex poles on the physical sheet. Even for $\gamma\neq 0$ the reaction matrix formalism is rather loose in this respect. We recall that the condition $(1-i\varrho_\pi q)(1-i\varrho_{\mathrm{K}}\varkappa)+\varrho_\pi\varrho_{\mathrm{K}}\gamma^2\neq 0$ for $z\geq m_\Sigma^2$ had to be added «by hand». This condition must be relaxed in the limit $\gamma\to 0$ in order to allow for the one-particle state described by the zero of $(1-i\varrho_{\mathrm{K}}\varkappa_0)$. If the mass of this particle is near the $\bar{\mathrm{K}}\mathcal{N}$' threshold, we may consider it as a bound state of the $\bar{\mathrm{K}}\mathcal{N}$' system; but this is a quantitative question [8].

So far we assumed that the coupling between the channels is sufficiently weak such that it is sensible to extrapolate to the limit of zero coupling. In this way it is possible to associate the resonance with a specific property of

[8] R. OEHME: *Nuovo Cimento*, **13**, 778 (1959). Plase note that the «deuteron» discussed on p. 786 of this paper should be considered as a model. For the actual deuteron the maximal range of the intrinsic elettromagnetic structure is determined by the three-pion state, but the composite structure is the same as in our model.

one of the isolated systems. However, as we have already pointed out in ref. [1], there is no reason why we should not have sharp resonances in the presence of a strong coupling between the channels [9]. In this case a given pair of resonance poles can generally not be related to a given channel. A change in the strength of the coupling or in some other part of the interaction will change the masses of the stable particles as well as the position of the resonance poles. The whole pattern of singularities can change in different ways corresponding to different models for the creation of the mass spectrum and the mutual interaction between the particles.

Poles of the amplitudes $F$, $K$ and $G$ in sheet III are due to zeros of the denominator

$$(1 + i\varrho_\pi \varphi)(1 + i\varrho_K \varkappa) + \varrho_\pi \varrho_K \gamma^2 . \tag{3.8}$$

In the case of weak coupling we take again the limit $\gamma \to 0$ and find

$$F^{III} \to F^{II} \to \frac{\varphi_0}{1 + i\varrho_\pi \varphi_0} , \tag{3.9}$$

$$K^{III} \to K^{IV} \to \frac{\varkappa_0}{1 + i\varrho_K \varkappa_0} . \tag{3.10}$$

A pair of resonance poles near the real axis for $s \geqslant (m_K + m_N)^2$ can be related, for $\gamma \to 0$, to resonance poles in the $\pi Y$ or the $\bar{K}N$ system [1].

* * *

We would like to thank Professor R. H. Dalitz for several interesting discussions.

[9] L. Fonda and R. G. Newton: *Ann. Phys.*, **10**, 490 (1960).

## RIASSUNTO (*)

Entro lo schema della teoria relativistica della dispersione la matrice di reazione (matrice $K$) viene considerata una funzione matriciale analitica. Dimostriamo che, nel caso di due canali accoppiati di due particelle, gli elementi di questa matrice sono funzioni regolari, salvo i tagli dovuti alle linee di branching dinamico delle ampiezze ed i possibili poli isolati. Esprimendo le ampiezze di reazione in funzione degli elementi della matrice di reazione, diventa possibile rappresentare la superficie riemanniana di queste ampiezze. Con l'ausilio del formalismo della matrice di reazione discutiamo la connessione fra risonanze e poli delle ampiezze nei foglietti secondari di Rieman. Prendiamo in considerazione la possibile connessione fra una risonanza ed uno stato legato.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. XX, N. 2 16 Aprile 1961

# Proposal of an Experiment on $\Sigma^{\pm}$-Decays.

PH. MEYER (*), J. PRENTKI and Y. YAMAGUCHI (**)

*CERN - Geneva*

(ricevuto il 9 Febbraio 1961)

**Summary.** — A discussion is given of specific experimental conditions under which the $s$ or $p$-wave nature of the decays $\Sigma^{\pm} \to n + \pi^{\pm}$ can be determined through a study of the neutron polarization. Besides being a check of the $|\Delta \boldsymbol{I}| = \frac{1}{2}$ rule this experiment gives a measure of the $\Sigma^{+}$ polarizations and therefore leads to the determination of the magnitude and sign of the asymmetry coefficient $\alpha^0$.

The experimental determination of the $s$ or $p$-wave nature of the decays

(1*a-b*) $$\Sigma^{+(-)} \to n + \pi^{+(-)}$$

using polarized $\Sigma$ and measuring the polarization of the decay neutron through its scattering on an appropriate analyser is discussed under *specific experimental conditions* which will be described in detail.

As is well known the knowledge of the $s$ or $p$ character is extremely important for the theory of weak interactions, and in particular provides one of the tests of the $|\Delta \boldsymbol{I}| = \frac{1}{2}$ rule (1).

Furthermore from the polarization of the neutron one can obtain the polarization of the $\Sigma$, $\boldsymbol{P}_{\Sigma}$, both in magnitude and sign. Therefore if in the same experiment the up-down asymmetry in the decay

(2) $$\Sigma^{+} \to p + \pi^0$$

(*) On leave from the Faculté des Sciences, Orsay, and Université de Bordeaux

(**) On leave from Osaka City University, Osaka.

(1) See *e.g.* M. GELL-MANN and A. ROSENFELD: *Ann. Rev. Nucl. Sci.*, **7**, 454 (1957).

is measured, one then obtains both the sign and magnitude of the asymmetry coefficient $\alpha^0$.

In the usual method one starts from unpolarized $\Lambda$'s or $\Sigma$'s and the longitudinal polarization of the proton is measured ([2]). Our proposal (based essentially on the fact that $\alpha^+ \simeq 0$) which might seem at first sight rather indirect (and even more difficult than the usual method because the polarization of the *neutron* is needed) seems however to have some substantial advantages which will be discussed further.

Let the $\Sigma^{\pm}$ be produced for instance through the reaction

$$\mathrm{K}^- + \mathrm{p} \to \Sigma^{\pm} + \pi^{\mp} . \tag{3}$$

By measuring the energy and angle of the $\pi$, one fixes the energy and angle of the produced $\Sigma$ and the direction of $\boldsymbol{P}_{\Sigma}$ ([3]) (perpendicular to the plane of production), though not its sign.

Now consider the decay of the $\Sigma$'s of definite energy and direction. It is well known ([4]) that the polarization of the nucleon is given by

$$\boldsymbol{P}_{N} = \frac{1}{1 - \alpha \hat{n} \boldsymbol{P}_{\Sigma}} \{ (\hat{n} \boldsymbol{P}_{\Sigma} - \alpha) \hat{n} + \beta (\hat{n} \times \boldsymbol{P}_{\Sigma}) + \gamma (\hat{n} \times \boldsymbol{P}_{\Sigma}) \times \hat{n} \} , \tag{4}$$

with

$$\alpha = \frac{2 \operatorname{Re} (A_S A_P^*)}{|A_S|^2 + |A_P|^2} , \tag{5a}$$

$$\beta = \frac{2 \operatorname{Im} (A_S A_P^*)}{|A_S|^2 + |A_P|^2} , \tag{5b}$$

$$\gamma = \frac{|A_S|^2 - |A_P|^2}{|A_S|^2 + |A_P|^2} , \tag{5c}$$

where $A_S$, $A_P$ are the amplitudes of the $s$, $p$ wave in the $\Sigma$-decay, $\boldsymbol{P}_N(\boldsymbol{P}_{\Sigma})$ are the polarizations of the nucleon ($\Sigma$) (in their own rest frame) and $\hat{n}$ is a unit vector in the direction of the nucleon momentum in the rest frame of the $\Sigma$. To a very good approximation $\beta = 0$ ([5]).

From previous experiments ([6]) it is known that $|\alpha^+ P_{\Sigma^+}| = 0.03 \pm 0.05$,

---

([2]) E. Boldt, H. S. Bridge, D. O. Caldwell and Y. Pal: *Phys. Rev. Lett.*, **1**, 256 (1958); R. Birge and W. Fowler: *Phys. Rev. Lett.*, **5**, 254 (1960).

([3]) All polarizations are defined in the rest system of the particle considered.

([4]) T. D. Lee and C. N. Yang: *Phys. Rev.*, **108**, 1645 (L) (1957).

([5]) From invariance under time reversal $\beta$ depends on the $\pi$-nucleon phase shifts $\delta_1$, $\delta_3$, $\delta_{11}$, $\delta_{31}$ which are small ($|\beta| < 0.1$).

([6]) See *e.g.*, J. W. Cronin: *Proc. of the* 1960 *Rochester Conference*, p. 590.

$|\alpha^0 P_{\Sigma^+}| = 0.75 \pm 0.17$ and $|\alpha^- P_{\Sigma^-}| = 0.02 \pm 0.05$, where the index 0, +, — refers to reactions (2), (1*a*) and (1*b*) respectively.

At this point it is important to make the following remarks:

1) The fact that $\alpha^0$ is known experimentally to be close to 1 does not enable one to make a strong statement about the value of $\gamma^0$.

2) In the opposite situation where $\alpha^+$ and possibly $\alpha^-$ are known to be close to 0 one can conclude that $|\gamma^\pm|$ are almost equal to 1. More precisely if $|\alpha| = \varepsilon$ then $|\gamma| \doteq 1 \pm \frac{1}{2}\varepsilon^2$ and if $|\alpha| = 1 - \varepsilon$ then $|\gamma| \doteq \sqrt{2\varepsilon}$, where $\varepsilon \ll 1$.

This means that in reaction (1*a*) one can take $\alpha^+ = 0$ and $\gamma^+ = \pm 1$ with the +(—) sign if the decay is through a pure $s(p)$ wave (7). It is precisely for this reason that an experimental determination of the neutron polarization in (1*a*) apart from deciding between $s$ or $p$ wave decay will give us at the same time a reliable measurement of $\boldsymbol{P}_{\Sigma^+}$, the knowledge of which obviously gives $\alpha^0$.

For the reasons mentioned above the study of the decay of polarized $\Sigma$ through (2) is not suitable for this purpose. Though $|\alpha^0| \simeq 1$ the presence of $\gamma^0$ in (4) prevents us from drawing similar conclusions from the measurement of the proton polarization as can be done from the neutron polarization (8). This is why it was necessary to use unpolarized $\Lambda$'s to get $\alpha^\Lambda$ from the polarization of the decay proton. Since it seems rather difficult to realize experimentally a suitable unpolarized sample it might be very useful to have different methods for the determination of $\alpha^0$. This situation which exists for $\Sigma$'s and not for $\Lambda$'s should in our opinion be taken advantage of.

In the case of the decay (1*b*) one cannot conclude rigorously from $|\alpha^- P_{\Sigma^-}| \simeq 0$ that $\alpha^- \simeq 0$ because $P_{\Sigma^-}$ itself might be small. However, recent experiments (9) show that $\alpha^-$ is very probably $\simeq 0$. If that is the case $\boldsymbol{P}_{\Sigma^-}$ can be obtained from $\boldsymbol{P}_{\mathcal{N}}$ as was discussed for case (1*a*).

Let us consider decay (1). In the case $\alpha = 0$ (4) reduces to

$$\boldsymbol{P}_{\mathcal{N}} = \boldsymbol{P}_{\Sigma}\,, \qquad \text{(for } s \text{ wave)}, \tag{6a}$$

$$\boldsymbol{P}_{\mathcal{N}} = -\boldsymbol{P}_{\Sigma} + 2(\hat{n}\boldsymbol{P}_{\Sigma})\hat{n} \qquad \text{(for } p \text{ wave)}. \tag{6b}$$

For later convenience we introduce the notation

$$\boldsymbol{P}_{\Sigma} = P_{\Sigma}\hat{\nu}\,, \tag{7}$$

where $\hat{\nu}$ is a unit vector normal to the production plane and $P_\Sigma$ is the magnitude of $\boldsymbol{P}_\Sigma$ *includig sign.*

---

(7) Since from experiment $|\alpha^+| < 0.13$ one gets $|\gamma^+| = 1$ within 1%.

(8) If however $|\alpha^0|$ turned out to be really 0.99 as suggested in ref. (9) by using the $|\Delta \boldsymbol{I}| = \frac{1}{2}$ rule the situation is not as unfavourable as stated above.

(9) See *e.g.* M. SCHWARTZ: *Proc. of the 1960 Rochester Conference*, p. 727.

Now let the polarized neutron from the $\Sigma$-decay be elastically scattered by a suitable analyser, for example carbon. If $\boldsymbol{k}_i$ and $\boldsymbol{k}_f$ are the initial and final neutron momenta in the laboratory system, and $\hat{\eta} = \boldsymbol{k}_i \times \boldsymbol{k}_f / |\boldsymbol{k}_i \times \boldsymbol{k}_f|$ is a unit vector normal to the scattering plane, the scattered intensity will be proportional to

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{lab}} \sim 1 + P_2(\theta, E_i)(\boldsymbol{P}_{\mathrm{N}} \cdot \hat{\eta}) , \tag{8}$$

where $\theta$ is the angle between $\boldsymbol{k}_i$ and $\boldsymbol{k}_f$ and $P_2(\theta, E_i)$ is a known function of $\theta$ and $E_i$, the neutron energy, and is characteristic of the chosen analyser ([10]).

From (7) it is evident that only the transverse polarization of the neutron in the laboratory system contributes to the left-right asymmetry. As a consequence of the $\Sigma$ velocity the longitudinal component of the neutron polarization in the rest system of the $\Sigma$ given by the second term of (6*b*) will have a transverse component in the laboratory system comparable to the contribution of the first term even in the non-relativistic approximation.

It can be shown that one finally obtains the following expressions:

i) in the case of an *s* wave

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{lab}} \sim 1 + P_2(\theta, E_i) P_\Sigma \sin\chi \cos\varphi ; \tag{9}$$

ii) in the case of a *p* wave

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{lab}} \sim 1 + P_2(\theta, E_i)\, P_\Sigma \cdot \tag{10}$$

$$\cdot \left\{ -\sin\chi \cos\varphi + 2A \frac{\sin\lambda^* \cos\mu}{\sin\chi} (\cos\lambda \cos\chi \cos\varphi + \sin\lambda \sin\mu \sin\varphi) \right\} ,$$

where

$$A = \frac{|v_\Sigma|}{|v_{\mathrm{n}}^*|} \left\{ \frac{1}{\sqrt{1 - v_\Sigma^2}} - \frac{|v_{\mathrm{n}}^*|}{|v_\Sigma|} \cos\lambda^* \left(1 - \frac{1}{\sqrt{1 - v_\Sigma^2}}\right) \right\} , \tag{11}$$

$$\cos\chi = \sin\lambda \cos\mu , \tag{12}$$

$\boldsymbol{v}_\Sigma$ is the velocity of the $\Sigma$ in the laboratory system, $\boldsymbol{v}_{\mathrm{n}}^*$ is the velocity of the decay neutron in the $\Sigma$ rest system ($|\boldsymbol{v}_{\mathrm{n}}^*| \simeq 0.2$, $c = 1$), $\lambda^*$ is the angle between $\boldsymbol{v}_{\mathrm{n}}^*$ and $\boldsymbol{v}_\Sigma$, while $\lambda$ and $\mu$ are the polar angles of $\boldsymbol{k}_i$ if $\boldsymbol{v}_\Sigma$ is taken as the $z$ axis and $\hat{\nu}$ as the $x$ axis, $\varphi$ is the angle between, $\hat{\eta}$ and the transverse component of $\hat{\nu}$ in the $(\boldsymbol{k}_i, \hat{\nu})$ plane.

Expressions (9) and (10) follow from a relativistic calculation and are exact.

---

([10]) For a plot of $P_2(\theta, E_i)$ in the case of n-$^{12}$C elastic scattering which is equal to $P_2(\theta, E_i)$ for p-$^{12}$C, see *e.g.* R. Birge and W. Fowler: ref. ([2]).

However, for a $\Sigma$ of about 300 MeV (kinetic energy) $|\boldsymbol{v}_n^*|/|v_\Sigma| \simeq \frac{1}{3}$ and sin $\lambda$ is small. In fact from the relativistic law of addition of velocities one gets

$$\operatorname{tg} \lambda = \frac{|v_n^*| \sqrt{1 - v_\Sigma^2}}{|v_\Sigma| + |v_n^*| \cos \lambda^*} \sin \lambda^* .$$

Expanding in powers of $|v_n^*|/|v_\Sigma|$ and keeping only terms up to the first order, expressions (9) and (10) become

i) for $s$ wave

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{lab}} \sim 1 + P_2(\theta, E_i) P_\Sigma \cos \varphi ; \tag{13}$$

ii) for $p$ wave

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{lab}} \sim 1 + P_2(\theta, E_i) P \cdot \cdot \left\{ - \cos \varphi + 2 \sin^2 \lambda^* \cos \mu \cos (\varphi - \mu) \left(1 - \sqrt{1 - v_\Sigma^2} \frac{|v_n^*|}{|v_\Sigma|} \cdot \cos \lambda^*\right)\right\} . \tag{14}$$

Discrimination between $s$ and $p$ wave can be made in a number of ways. One of the most suitable seems to be the following. From (13) it is seen that one gets for $s$ wave a left-right asymmetry with respect to the $(\boldsymbol{k}_i, \hat{\nu})$ plane. On the other hand, in the case of a $p$ wave the first term in the bracket of (14) gives a left-right asymmetry and the second term contributes both a left-right asymmetry with respect to the $(\boldsymbol{k}_i, \hat{\nu})$ plane and an « up-down » asymmetry with respect to the plane defined by $\boldsymbol{k}_i$ and $(\boldsymbol{k}_i \times \hat{\nu})$. Therefore by choosing $\lambda^*$ and $\mu$ in an appropriate manner one can obtain a definite *left-right* asymmetry in the case of $s$ wave and *no left-right* asymmetry in the case of $p$ wave, while at the same time one gets *no* « *up-down* » asymmetry in the case of $s$ wave and a definite « *up-down* » asuymmetry in the case of $p$ wave.

More specifically, let us take $\lambda^* = \pi/2$ and $\mu = \pi/4$ [11]. Then (13) and (14) lead to [12]:

[11] For discussion's sake we are making several restrictions on the different angles involved. However, it is easily seen that these restrictions can be partially relaxed as long as the discrimination between $s$ and $p$ wave remains sufficiently unambiguous.

[12] We define as usual

$$\text{« } L/R \text{ »} = \frac{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(\varphi = 0) - \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(\varphi = \pi)}{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(\varphi = 0) + \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(\varphi = \pi)} \quad \text{and} \quad \text{« } U/D \text{ »} = \frac{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\left(\varphi = \frac{\pi}{2}\right) - \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\left(\varphi = \frac{3\pi}{2}\right)}{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\left(\varphi = \frac{\pi}{2}\right) + \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\left(\varphi = \frac{3\pi}{2}\right)} .$$

Note that the choice of axis is such that $\varphi = \pi/2$ corresponds to $\hat{\eta}$ in the direction of $k_i \times \hat{\nu}$.

i) for $s$ wave

$$(15)\qquad \begin{cases} \langle\!\langle\, U/D \,\rangle\!\rangle = 0 & (+0)\,, \\ \langle\!\langle\, L/R \,\rangle\!\rangle = P_2(\theta, E_i)P_\Sigma\,, & (-0.02\,P_2 P_\Sigma); \end{cases}$$

ii) for $p$ wave

$$(16)\qquad \begin{cases} \langle\!\langle\, U/D \,\rangle\!\rangle = P_2(\theta, E_i)P_\Sigma & (-0.02\,P_2 P_\Sigma)\,, \\ \langle\!\langle\, L/R \,\rangle\!\rangle = 0 & (-0.03\,P_2 P_\Sigma)\,. \end{cases}$$

The values in parenthesis (calculated for $|v_\Sigma| = 0.6$, $v_n^*/v_\Sigma = \frac{1}{3}$) are the higher order corrections obtained from the exact expressions (9) and (10). Once $\langle\!\langle U/D \rangle\!\rangle$ and $\langle\!\langle L/R \rangle\!\rangle$ have been measured, from (15) and (16) the following conclusions can be drawn:

*a*) If $\langle\!\langle U/D \rangle\!\rangle = 0$ and $\langle\!\langle L/R \rangle\!\rangle \neq 0$ then $\Sigma$ decays through an $s$ wave and $P_\Sigma \neq 0$.

*b*) If $\langle\!\langle U/D \rangle\!\rangle \neq 0$ and $\langle\!\langle L/R \rangle\!\rangle = 0$ then $\Sigma$ decays through a $p$ wave and $P_\Sigma \neq 0$.

*c*) If $\langle\!\langle U/D \rangle\!\rangle = 0$ and $\langle\!\langle L/R \rangle\!\rangle = 0$ then $P_\Sigma = 0$.

*d*) If both $\langle\!\langle U/D \rangle\!\rangle$ and $\langle\!\langle L/R \rangle\!\rangle$ differ substantially from zero then our assumption $\alpha = 0$ is not valid and $P_\Sigma^- \neq 0$. For $\Sigma^{\pm} \to n + \pi^{\pm}$, $\alpha \neq 0$ can happen only in the $\Sigma^-$ decay since it is already known that $\alpha^+$ is very close to 0.

Case *d*) would mean therefore that the $|\Delta \boldsymbol{I}| = \frac{1}{2}$ rule would not be valid.

If situation *a*) or *b*) has been found experimentally, one can determine not only the magnitude but also the sign of $P_\Sigma$ using formula (15) or (16).

It is to be noted that our proposal seems well adapted to a counter experiment, the higher counting rate compensating for the angular restrictions required ([11]) which is rather similar to that of a triple scattering experiment.

If the proposed experiment is performed both for decay (1*a*) and (1*b*) one has an obvious check of the $|\Delta \boldsymbol{I}| = \frac{1}{2}$ rule. It is well known that an $s$ wave in $\Sigma^+$-decay and a $p$ wave in $\Sigma^-$-decay, or vice versa, combined with other experimental information on $\Sigma$-decays, constitute in fact a proof of the $|\Delta \boldsymbol{I}| = \frac{1}{2}$ rule. The same $s$ (or $p$) wave nature in both $\Sigma^{\pm}$ decays contradicts the $|\Delta \boldsymbol{I}| = \frac{1}{2}$ rule, as well as case *d*) discussed above.

In conclusion, it has been seen that under specified conditions the study of the neutron polarization in $\Sigma^{\pm}$-decay can provide us not only with the knowledge of the $s$ and $p$ wave nature of the decay which is important *per*

*se* for the theory of weak interactions, but also with an unambiguous measure of the polarization (including sign) of the parent $\Sigma$. This, combined with the knowledge of $\alpha^0 P_{\Sigma^+}$ from up-down asymmetry, determines the magnitude *and sign* of $\alpha^0$. It seems to us that this method of obtaining $\alpha^0$ through a direct measurement of $P_{\Sigma^+}$ starting with polarized $\Sigma^+$'s and taking advantage of the effective conservation of parity in the neutron decay mode has perhaps some advantages over the procedure (which was used in $\Lambda^0$-decay) consisting in measuring the longitudinal polarization of the proton coming from unpolarized $\Sigma^+$'s.


## RIASSUNTO (*)

Si discutono le condizioni sperimentali specifiche alle quali si può determinare la natura di onda $s$ o $p$ dei decadimenti $\Sigma^{\pm} \to n + \pi^{\pm}$, tramite uno studio della polarizzazione del neutrone. Oltre a costituire una verifica della regola $|\Delta I| = \frac{1}{2}$, questo esperimento dà una misura della polarizzazione dei $\Sigma^-$ e quindi porta alla determinazione in grandezza e segno del coefficiente di asimmetria $\alpha^0$.


(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. XX, N. 2 16 Aprile 1961

# The Decay of Negative $\pi$-Mesons Stopped in Light Elements and Insulators.

G. Culligan, D. Harting, N. H. Lipman, L. Madansky (*) and G. Tibell

*CERN - Geneva*

(ricevuto il 13 Febbraio 1961)

**Summary.** — Positive and negative beams of 170 MeV/c $\pi$-mesons, from which the $\pi$-mesons were selected by time-of-flight, were stopped in Be, $B_4C$, Teflon and Al targets. The ratio of the numbers of electrons produced in the negative and positive beams, by $\pi$-$\mu$-e decay, was determined for each of these materials. Assuming that no decay of negative $\pi$-mesons occurs in the Al target, an upper limit of one per cent is found for the decay probability of negative $\pi$-mesons stopped in either Be, $B_4C$ or Teflon.

## Introduction.

It has been observed by Stearns and Stearns [1] that the $K$ and $L$ X-ray yields from light $\mu$ and $\pi$ mesonic atoms are much lower than expected [2]. This is difficult to explain. One suggestion is that some mesons in falling to the ground state do not emit an X-ray at every transition but transfer energy to Auger electrons. However, theoretical predictions [3] of the Auger rate, confirmed by experiments in nuclear emulsions [4], cannot explain the results

(*) National Science Foundation Senior Postdoctoral Fellow on leave from Johns Hopkins University 1960-1961, Baltimore, Md.

(1) M. B. Stearns and M. Stearns: *Phys. Rev.*, **105**, 1573 (1957).

(2) M. A. Ruderman: *Phys. Rev.*, **118**, 1632 (1960).

(3) G. R. Burbidge and A. H. de Borde: *Phys. Rev.*, **89**, 189 (1953); A. H. de Borde: *Proc. Phys. Soc. (London)*, **67**, 57 (1954).

(4) W. F. Fry: *Phys. Rev.*, **85**, 676 (1952). A. Pevsner, L. Madansky, R. Strand and T. Toohig: *Proceedings of the 1960 Annual Rochester Conference*, p. 547.

of STEARNS and STEARNS. Another possibility is that some mesons do not cascade to the ground state within the predicted times, which are short compared with the $\pi$-meson lifetime, but are trapped in a high-energy state until they decay.

There is no reason, theoretically, to expect that mesons can be trapped in the materials used by STEARNS and STEARNS, though calculations by HUBY [5] indicate that a meson might be trapped in an insulator. There is some experimental evidence that this happens. FRY and WHITE [6] observed 18 decay events for 40 000 $\pi^-$-mesons stopping in emulsion, which is higher than expected from decay in flight in the emulsion, and could possibly result from some trapping mechanism.

The low mesic X-ray yield for light elements, coupled with the results of FRY and WHITE and the calculation of HUBY, encouraged us to search in a systematic way for the trapping of negative $\pi$-mesons. The technique consisted in looking for the characteristic growth curve of the electrons from the $\pi$-$\mu$-e decay sequence, when negative mesons were brought to rest in the target material.

We considered boron carbide and beryllium to be suitable low $Z$ target materials exhibiting a low mesonic X-ray yield. Teflon was chosen as a representative insulator, and aluminium as a material in which there is no reason to expect mesons to be trapped. The results in aluminium serve as a check on the background and systematic errors in the experiment.

## 1. – Experimental arrangement.

The experimental arrangement is illustrated in Fig. 1. The 170 MeV/$c$ $\pi$-meson beam from the CERN synchro-cyclotron was used for the experiment. The $\pi$-mesons were moderated by 10 g/cm$^2$ carbon and 2 g/cm$^2$ copper, and were brought to rest in the target which was inclined at 45° to the incident beam. An incoming $\pi$-meson was identified by its time-of-flight between counters 1 and 2 which were separated by 7 m.

Counters 3 and 4 were used to detect electrons emitted from the target, the time interval between the incident $\pi$-meson passing through counter 2 and an electron passing through counter 3 being recorded with a timesorter.

Table I contains the dimensions of the various targets used. As far as possible, the targets were constructed to have equal stopping power and geometry.

(5) R. HUBY: *Phil. Mag.*, **40**, 685 (1949).
(6) W. F. FRY and G. R. WHITE: *Phys. Rev.*, **93**, 1427 (1954).

TABLE I. – *Dimensions of the targets used.*

| Target material | Dimensions in cm | Thickness in the beam (g/cm²) |
|---|---|---|
| Beryllium | 19.1 × 12.7 × 3.2 | 7.9 |
| Boron carbide ($B_4C$) | 18.0 × 12.0 × 4.0 | 6.5 |
| Teflon ($CF_2$) | 19.1 × 12.7 × 2.4 | 7.4 |
| Aluminium | 19.0 × 12.7 × 2.1 | 7.8 |

Measurements were made using both the positive and negative π-meson beams. After correction for background, the ratio between the number of

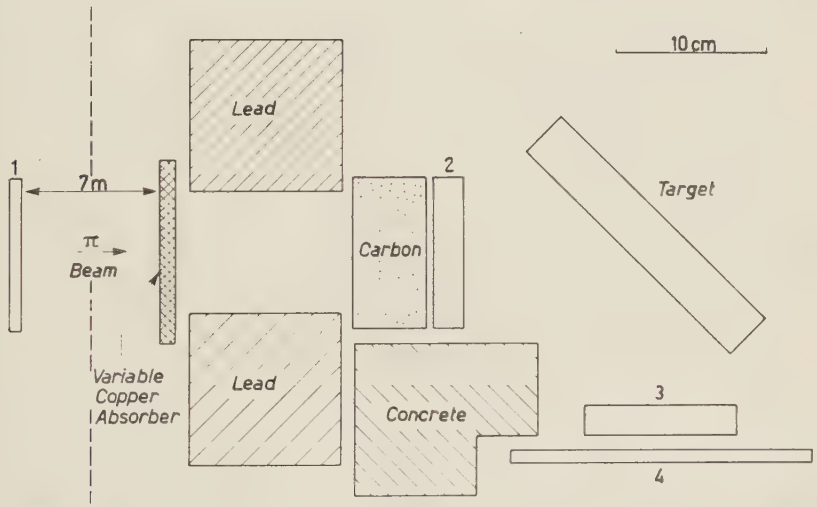


Fig. 1. Schematic diagram of the experimental arrangement. Counter dimensions (cm³) 1: 10 × 10 × 0.8; 2: 10 × 10 × 2; 3: 10 × 10 × 2; 4: 15 × 2 × 0.8.

events detected per incident negative meson and the number of events detected per incident positive meson, gives an estimate of the probability of trapping for negative π-mesons.

## 2. – Electronic apparatus.

Fig. 2 is a block diagram of the electronic apparatus. A particle passing through counters 1 and 2 triggered coincidence circuit *A* (resolving time ~ 20 ns) which then fed a gating pulse to the time-to-pulse-height converter *B*.

This time-to-pulse-height converter had a time range of 12 ns and a resolving time of 1 ns (7). The start pulse for $B$ was taken from counter 1 and the stop pulse from counter 2. Fig. 3 is a typical output pulse-height spectrum from $B$.

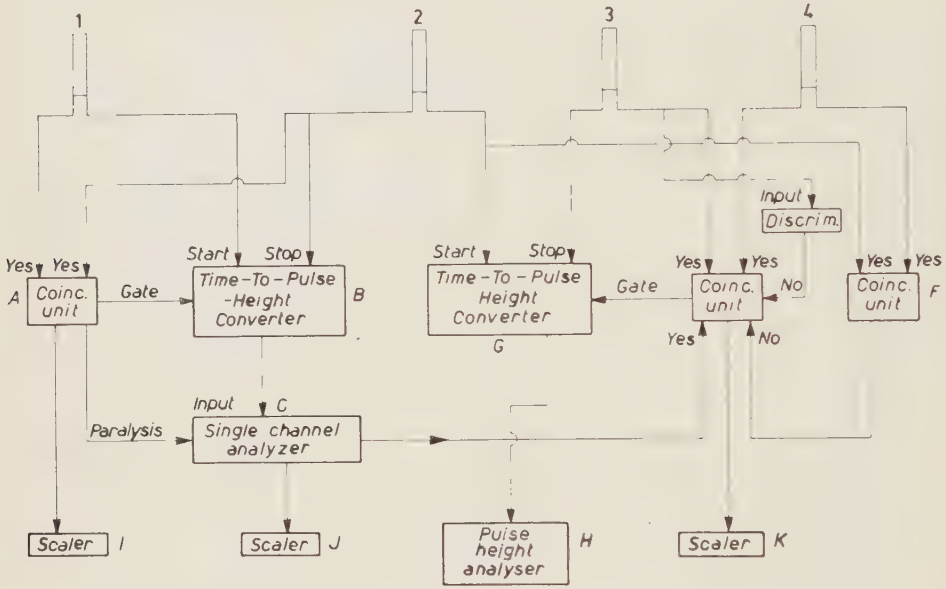


Fig. 2. – Block diagram of the electronic apparatus.

The output pulses from $B$ were fed to a single channel analyser $C$ which was set to accept pulses corresponding to the flight time of a $\pi$-meson. The dotted lines in Fig. 3 indicate the range of pulse heights accepted by $C$.

The output pulse from the single channel analyser $C$ together with pulses from the electron counters 3 and 4 were fed to the triple coincidence circuit $D$, the delay cables being chosen to bring counters 3 and 4 into coincidence (resolving time 20 ns). The length and delay of the pulses from $C$ were arranged such that an electron detected within 250 ns of the incident $\pi$-meson, triggered the coincidence circuit $D$, which then fed a gating pulse to the time-to-pulse-height converter $G$. The pulses from $A$, $C$ and $D$ were counted on the scalers $I$, $J$ and $K$, the output from $C$ serving as a monitor.

The time-to-pulse-height converter $G$ had a time range of 200 ns and a resolving time of 1 ns. The start pulse for $G$ was taken from counter 2 and the stop pulse from counter 3, and the output pulses from $G$ were recorded on the pulse-height analyser $H$. In the positive beam, the time spectrum

(7) G. CULLIGAN and N. H. LIPMAN: *Rev. Sci. Instr.*, **31**, 1209 (1960).

from $G$ consisted of a « zero time » peak, followed by the growth curve of the positrons from the π-μ-e decay sequence.

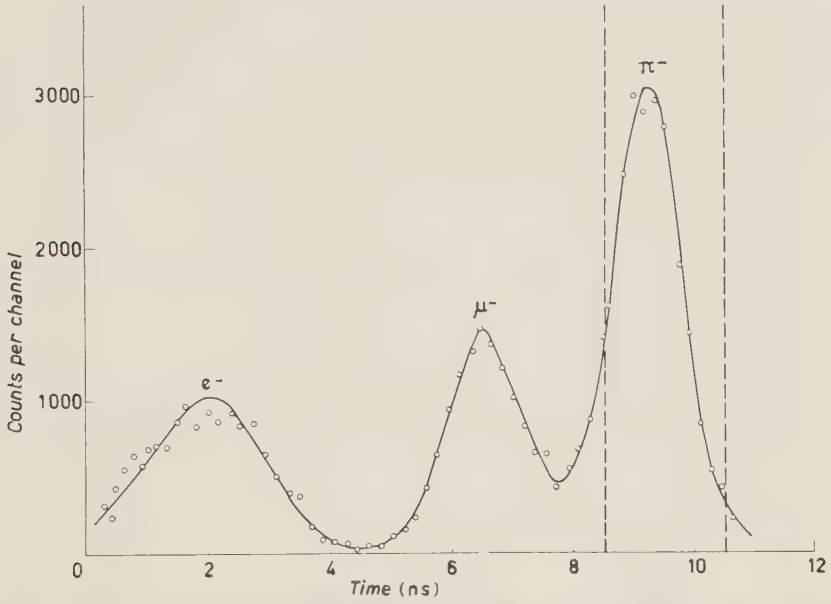


Fig. 3. – A typical pulse-height spectrum from the time-to-pulse-height converter $B$ for the negative beam.

The zero time peak was caused by π-mesons which interacted in the target, giving instantaneous events in the electron telescope. The half width of this zero time peak was 2 ns and, hence, it did not interfere significantly with the experiment. Events in the time spectrum that do not correspond to electrons from a π-μ-e decay in the target form a background which has to be subtracted. Part of this background is difficult to remove, and is caused by π-mesons decaying in flight into μ-mesons which then stop in the target. The resulting decays gave rise to an almost constant counting rate over the 200 ns range. The remaining background was of a random nature, and was reduced by the following measures:

*a*) We required that the pulse height in counter 3 corresponded to a minimum ionizing particle. This was done by feeding an output from 3 into a discriminator $E$, the output from which vetoed the action of coincidence circuit $D$.

*b*) Events were vetoed in which the pulse in 4 came within 5 ns of a pulse in 2. The pulses from 2 and 4 were fed to the fast coincidence circuit $F$,

the output from which vetoed coincidence circuit $D$. This measure reduced the effective counting rate in the electron telescope and, hence, also the random background.

*c*) A paralysis pulse was fed into the single channel analyser $C$ from the coincidence circuit $A$. This pulse prevented $C$ from operating in a time interval from 250 ns to 10 μs after an incoming particle gave a (1, 2) coincidence. The paralysis pulse was extended if there were two pulses from coincidence circuit $A$ within 10 μs. This requirement removed random events produced by incident π-mesons together with decay electrons from previous beam particles.

### 3. – Results.

With the experimental arrangement described above, both positive and negative pion decays were studied for the four different target materials. In Fig. 4, the results are presented for the total of all the runs made with posi-

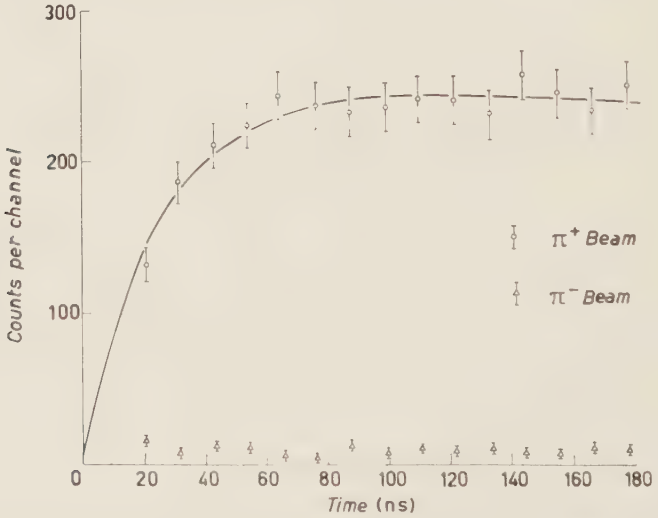


Fig. 4. – Pulse-height spectra from the time-to-pulse-height converter $G$ for the positive and negative beams respectively.

tive and negative π-mesons respectively. The ordinates are normalized to the same number of incoming pions (counts from $C$).

The full line drawn in Fig. 4 is a best fit to the results obtained in the positive beam of the function:

$$A \exp [-\lambda_\mu t] - B \exp [-\lambda_\pi t] ,$$

where $\lambda_\mu$ is the decay constant of the μ-meson and $\lambda_\pi$ that of the π-meson. In fitting this curve, we have assumed that the results in the negative beam give an upper limit to the almost constant background $(A - B)$ present in the positive beam. The best two-parameter fit of $A$ and $\lambda_\pi$ gives a value of $(27 \pm 3)$ ns for the pion mean life, which agrees well with the most accurate measurement to date of $(25.46 \pm 0.32)$ ns ([8]).

Table II gives the raw data obtained in this experiment. It contains the monitor counts (scaler $J$), the events recorded on the pulse-height analyser $H$ from 15 ns to 180 ns after the pulse in counter 2, the target material and the

TABLE II. – *Results obtained with each target. The monitor counts give the number of incoming π-mesons; the events are the number of counts in the time spectrum from* 15 ns *to* 180 ns *after zero time.*

| Target material | $\pi^+$ Beam | | $\pi^-$ Beam | |
|---|---|---|---|---|
| | Monitor counts | Events | Monitor counts | Events |
| Beryllium | 387900 | 720 | 503700 | 40 |
| Boron carbide ($B_4C$) | 138000 | 196 | 163700 | 12 |
| Teflon ($CF_2$) | 992600 | 1681 | 1505100 | 116 |
| Aluminium | 400000 | 804 | 450500 | 41 |
| Total | 1918500 | 3401 | 2622900 | 209 |
| No target | — | — | 200000 | 2 |

beam used. Table III is an analysis of the numbers listed in Table II. It is seen that the ratio of events in the positive beam to events in the negative beam is independent of the target material within the accuracy of the experiment.

TABLE III. – *Analysis of results obtained with various target materials.*

| Target material | Events/$10^5$ monitor counts $\pi^+$ beam | Events/$10^5$ monitor counts $\pi^-$ beam | Ratio $(\pi^-/\pi^+)$ (%) |
|---|---|---|---|
| Beryllium | $186 \pm 6.9$ | $7.9 \pm 1.3$ | $4.3 \pm 0.7$ |
| Boron carbide ($B_4C$) | $142 \pm 10.1$ | $7.3 \pm 2.1$ | $5.2 \pm 1.5$ |
| Teflon ($CF_2$) | $169 \pm 4.1$ | $7.7 \pm 0.7$ | $4.6 \pm 0.4$ |
| Aluminium | $201 \pm 7.1$ | $9.1 \pm 1.4$ | $4.5 \pm 0.7$ |
| Total | $177 \pm 3.0$ | $8.0 \pm 0.6$ | $4.5 \pm 0.3$ |
| No target | — | $1.0 \pm 0.7$ | — |

([8]) T. ASHKIN, T. FAZZINI, G. FIDECARO, Y. GOLDSCHMIDT-CLERMONT, N. H. LIPMAN, A. W. MERRISON and H. PAUL: *Nuovo Cimento*, **16**, 490 (1960).

A calculation of the number of pions, which, having triggered the time-of-flight apparatus, decay into $\mu$-mesons before stopping in the target, can account for the number of events observed in the negative beam.

The ratio of $\pi^-/\pi^+$ events in aluminium may be used as a measure of the background from all sources. If this is subtracted from the respective ratios in the other three materials, the following branching ratios of trapping-to-absorption are obtained:

| | |
|---|---|
| Beryllium | $(-0.2 \pm 1.0)\,\%$ |
| Boron Carbide ($B_4C$) | $(0.6 \pm 1.7)\,\%$ |
| Teflon ($CF_2$) | $(0.0 \pm 0.8)\,\%$. |

We conclude that, within the statistics of this experiment, trapping does not occur in beryllium, boron carbide and teflon. The deficiency of about 30% which is observed in the yield of mesonic $K$ X-rays from beryllium and boron is therefore not due to trapping of the meson in a high atomic orbit. The results also indicate that the mechanism postulated by HUBY, by which mesons might be trapped in insulators, does not occure in Teflon with a probability larger than 1%.

* * *

The authors wish to thank Prof. J. C. KLUYVER and Mr. D. WIEGANDT for help during the runs. The technical help of Mr. V. BECK and Mr. R. SUGDEN and the co-operation of the CERN synchro-cyclotron crew is gratefully acknowledged.

## RIASSUNTO (*)

Abbiamo arrestato su bersagli di Be, $B_4C$, Teflon ed Al fasci positivi e negativi di mesoni $\pi$ di 170 MeV/c, i cui mesoni $\pi$ erano stati selezionati in base al tempo di volo. Il rapporto numerico degli elettroni prodotti per decadimento $\pi$-$\mu$-e nei fasci negativi e positivi è stato determinato per ciascuna di queste sostanze. Supponendo che nel bersaglio di Al non abbia luogo alcun decadimento di mesoni $\pi$ negativi, abbiamo trovato che l'1% è il limite superiore per la probabilità di decadimento dei mesoni $\pi$ negativi arrestati in Be, $B_4C$, o Teflon.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. XX, N. 2 16 Aprile 1961

# Investigation of the Possibility of a $\pi$-$\Lambda$ Resonance in the Case of Different Parities of $\Lambda$ and $\Sigma$.

F. Duimio (*) and G. Wolters

*CERN - Geneva*

(ricevuto il 18 Febbraio 1961)

**Summary.** — In view of the recently obtained experimental information concerning energy and total angular momentum of the $\pi$-$\Lambda$ resonant state, an analysis is made of pion-hyperon scattering based on the assumption of different parities of $\Lambda$ and $\Sigma$. Since the resonance occurs in the low energy region, a fixed source field theoretic model is used. Confidence in this model is given by the fact that it leads to a good description of $\pi$-$\mathcal{N}$ low energy scattering. It is found that the present data are in better agreement with our basic assumptions than with global symmetric assumptions.

---

The existence of a resonant state in pion-hyperon scattering has recently been confirmed experimentally (1). Due to its importance in the physics of strongly interacting particles, various theoretical previsions on its nature and its implications were attempted before its experimental discovery (2). These were based on various assumptions on the nature and the symmetries of the elementary interactions. In particular, Amati, Stanghellini and Vitale

(*) On leave from Istituto di Fisica dell'Università di Milano, Italy.

(1) M. Good: *Proc. X High Energy Conference Rochester* (1960); M. Alston, L. Alvarez, Ph. Eberhard, M. L. Good, W. Graziano, H. K. Ticho and S. G. Wojcicki: *Phys. Rev. Lett.*, **5**, 520 (1960).

(2) D. Amati, A. Stanghellini and B. Vitale: *Nuovo Cimento*, **13**, 1143 (1959); *Phys. Rev. Lett.*, **5**, 524 (1960); M. Nauenberg: *Phys. Rev. Lett.*, **2**, 351 (1959); R. Capps and M. Nauenberg: *Phys. Rev.*, **118**, 593 (1960); R. Capps: *Phys. Rev.*, **119**, 1753 (1960).

studied the problem, applying the fixed-source theory, in the no-crossing, one meson approximation [3]. From the beginning, global symmetry was postulated, at least as a guiding principle, but possible small differences of the coupling constants $f_\Lambda$ and $f_\Sigma$ were taken into account, along with the $\Lambda$-$\Sigma$ mass difference. These authors predicted two resonant states for $J=\frac{3}{2}$, with $T=1$ and $=2$, respectively. However, at present it seems that the observed resonance is characterized by $J=\frac{1}{2}$. Such a resonance can only be explained by these authors taking a large difference between $f_\Lambda$ and $f_\Sigma$. This means a strong deviation from global symmetry. In such a situation one should introduce two cut-off parameters into the theory instead of one.

Moreover, the possibility of opposite parities of $\Lambda$ and $\Sigma$ has to be taken into account, as one of the simplest ways to abandon global symmetry.

In this paper we study the consequences of the hypothesis of opposite parities, using the same approach.

Although it certainly does not provide for a complete description of the scattering processes, it will at least lead to valuable qualitative understanding of the scattering in the low-energy region.

The $\Lambda$-parity is assumed to be $+1$. The interaction Hamiltonian is $H_{sc}+H_{pv}$ with

$$H_{pv}=\sqrt{4\pi}f_\Sigma\sum_{k,\alpha,m}C(k)\sigma_m[a_{\alpha,m}(k)T_\Sigma^\alpha+a_{\alpha,m}^+(k)T_\Sigma^{\alpha+}]\,,$$

(units $\hbar=c=\mu=1$),

$$H_{sc}=\sqrt{4\pi}f_\Lambda\sum_{k,\lambda}D(k)[a_\lambda(k)+a_\lambda^+(k)]T_\Lambda^\lambda\,,$$

where $a_\lambda^+(k)$ is the creation operator for a pion with momentum $k$, isotopic spin index $\alpha$, in an $s$-wave, while $a_{\lambda,m}^+(k)$ is the creation operator for a pion with momentum $k$, isotopic spin index $\alpha$, in a $p$-wave, $m$ specifying the third component of its orbital angular momentum ($m=1,2,3$),

$$D(k)=\frac{kv(k)}{\sqrt{\pi\omega}}\,,$$

($v(k)$ = cut-off function).

$$C(k)=\frac{k^3v(k)}{\sqrt{3\pi\omega}}\,,$$

[3] M. FRIEDMAN, T. LEE and R. CHRISTIAN: *Phys. Rev.*, **100**, 1496 (1955); R. STROFFOLINI: *Phys. Rev.*, **104**, 1146 (1956); B. BOSCO, S. FUBINI and A. STANGHELLINI: *Nucl. Phys.*, **10**, 663 (1959).

$T^{\alpha}_{\Lambda}$ and $T^{\alpha}_{\Sigma}$ are analogous to the $\tau$-operators in the pion-nucleon interaction. They are explicitly given by:

$$T^{\alpha}_{\Lambda} = \Sigma^{+}_{\alpha} \Lambda + \Lambda^{+} \Sigma_{\alpha} ,$$

$$T^{\alpha}_{\Sigma} = \varepsilon_{\alpha\beta\gamma} \Sigma^{+}_{\beta} \Sigma_{\gamma} ,$$

$\varepsilon_{\alpha\beta\gamma}$ = Levi-Civita tensor .

The scattering states are classified according to the total isotopic spin $T$, angular momentum $J$ and parity $P$. In the one meson approximation they are represented by the general formula:

$$|\pi_q Y\rangle = \sum_{q''} [a^{+}_{q''} |\Lambda\rangle X_1(q, q'') + a^{+}_{q''} |\Sigma\rangle X_2(q, q'')] + |\Lambda\rangle C_0 + |\Sigma\rangle C_1 ,$$

where $q$ comprises all relevant indices.

The $\Lambda$-$\Sigma$ mass difference $\Delta$ is taken into account explicitly. The other parameters, on which our results will depend, are the renormalized coupling constant $f_{\Lambda}$ and the product $f^2_{\Sigma} I$ with renormalized coupling constant $f_{\Sigma}$, and $I$ depending on one cut-off.

The various coefficients $X$ and $C$ are determined, using a well-known variational procedure ([2,3]). The $X$-functions must satisfy integral equations (or systems of coupled integral equations), which can be solved in an elementary way, using given boundary conditions, which describe the initial and final states of the scattering problems. The possibility of a resonance is determined by the vanishing of the denominators of the $X$-functions.

TABLE I.

| $T$ | $J$ | $P$ | state vector | denominator |
|---|---|---|---|---|
| 0 | $\frac{1}{2}$ | + | $\|\Lambda\rangle + \|\pi\Sigma\rangle_s$ | $\begin{vmatrix} 1+3f^2_{\Lambda}\omega\varrho^{+}_{+}(\omega-\Delta) & -f^2_{\Lambda}(\omega-2\Delta)\varrho^{-}_{+}(\omega-\Delta) \\ 3f^2_{\Lambda}\omega\varrho^{-}_{+}(\omega-\Delta), & 1-f^2_{\Lambda}(\omega-2\Delta)\varrho^{-}_{-}(\omega-\Delta) \end{vmatrix}$ |
| | | − | $\|\pi\Sigma\rangle_p$ | $1-2f^2_{\Sigma}(\omega-\Delta)I$ |
| | $\frac{3}{2}$ | + | — | — |
| | | − | $\|\pi\Sigma\rangle_p$ | $1+4f^2_{\Sigma}(\omega-\Delta)I$ |
| 1 | $\frac{1}{2}$ | + | $\|\pi\Lambda\rangle_p + \|\pi\Sigma\rangle_s$ | $1-6f^2_{\Lambda}f^2_{\Sigma}\omega^3 I\varrho^{+}_{+}(\omega-\Delta)$ |
| | | − | $\|\Sigma\rangle + \|\pi\Lambda\rangle_s + \|\pi\Sigma\rangle_p$ | $\begin{vmatrix} 1-f^2_{\Lambda}(\omega+\Delta)\varrho^{+}_{+}(\omega) & f^2_{\Lambda}(\omega-\Delta)\varrho^{-}_{+}(\omega) & \sqrt{2}f_{\Lambda}f_{\Sigma}(\omega-\Delta)\varrho^{-}_{+}(\omega) \\ -f^2_{\Lambda}(\omega+\Delta)\varrho^{-}_{+}(\omega) & 1+f^2_{\Lambda}(\omega-\Delta)\varrho^{-}_{-}(\omega) & \sqrt{2}f_{\Lambda}f_{\Sigma}(\omega-\Delta)\varrho^{-}_{-}(\omega) \\ 0 & 3\sqrt{2}f_{\Lambda}f_{\Sigma}(\omega-\Delta)I & 1+7f^2_{\Sigma}(\omega-\Delta)I \end{vmatrix}$ |
| | $\frac{3}{2}$ | + | $\|\pi\Lambda\rangle_p$ | — |
| | | − | $\|\pi\Sigma\rangle_p$ | $1-2f^2_{\Sigma}(\omega-\Delta)I$ |
| 2 | $\frac{1}{2}$ | + | $\|\pi\Sigma\rangle_s$ | $1+f^2_{\Lambda}(\omega-2\Delta)\varrho^{-}_{+}(\omega)$ |
| | | − | $\|\pi\Sigma\rangle_p$ | $1+f^2_{\Sigma}(\omega-\Delta)I$ |
| | $\frac{3}{2}$ | + | — | — |
| | | − | $\|\pi\Sigma\rangle_p$ | $1-2f^2_{\Sigma}(\omega-\Delta)I$ |

Omitting the tedious calculations which are necessary to arrive at expressions for these denominators, the final results are given in table I for all scattering states with the following definitions:

$$I = \frac{1}{3\pi}\int \mathrm{d}\omega' \, \frac{k'^3 v^2(k')}{\omega'^2(\omega' - \omega)} ,$$

$$\varrho^{\pm}_{\pm}(\omega) = \frac{1}{\pi}\int \mathrm{d}\omega' \, \frac{k' v^2(k')}{(\omega' \pm \Delta)^2(\omega' - \omega)} ,$$

$$\varrho^{-}_{+}(\omega) = \frac{1}{\pi}\int \mathrm{d}\omega' \, \frac{k' v^2(k')}{(\omega'^2 - \Delta^2)(\omega' - \omega)} .$$

Before examining these results, we want to stress two peculiar aspects, due to the introduction of opposite $\Lambda$-$\Sigma$ parities:

*a*) The doubling of the number of possible states as a consequence of parity conservation, makes the one meson approximation rather poor. For example in the state $T=1$, $J=\frac{3}{2}$, $P=+$ no scattering occurs.

*b*) The presence of *s*-waves leads to integrals of the type $\varrho(\omega)$ which depend strongly on $\omega$ in the low-energy region, in contrast to integrals of the type $I(\omega)$, for which the effective range approximation is justifiable. However, the $\varrho$'s practically do not depend on the cut-off. This eliminates one of the two cut-off parameters in the theory. This can be considered as a typical advantage of the hypothesis of opposite parities, within the framework of our method.

Before discussing the main features of the pion-hyperon scattering in our model, as far as resonances are concerned, we list the most important results in the tables II and III.

Table II.

| $J$ | $T$ | $P$ | possible resonant state |
|---|---|---|---|
| $\frac{1}{2}$ | 0 | — | $X_0^*(\Sigma)$ |
| $\frac{1}{2}$ | 1 | + | $Y^*_{(+)}(\Lambda, \Sigma)$ |
| $\frac{1}{2}$ | 1 | — | $Y^*_{(-)}(\Lambda, \Sigma)$ |
| $\frac{3}{2}$ | 1 | — | $X_1^*(\Sigma)$ |
| $\frac{3}{2}$ | 2 | — | $X_2^*(\Sigma)$ |

Although one might expect that the state $T=0$, $J=\frac{1}{2}$, $P=+$ also would give a resonance, it can be proved that in an energy region, much larger than $1.5 < \omega < 2.4$, the denominator can only vanish for $f^2_\Lambda < 0$. Then one is left

TABLE III.

| possible resonant state | $\omega$ | $f_\Lambda^2$ | $f_\Sigma^2 I$ |
|---|---|---|---|
| $X_i^*(\Sigma)$ | $1 < \omega < 2.5$ | — | $\geqslant 0.25$ |
| $Y_{(+)}^*(\Lambda, \Sigma)$ | $1.5 < \omega < 2.5$ | $> 1.5$ | $\leqslant 0.1$ |
| $Y_{(-)}^*(\Lambda, \Sigma)$ | $1 < \omega < 2.4$ | $> 1.5$ | — |

with five possible resonant states, of which the three states denoted by $X_i^*(\Sigma)$ have the same location on the $\omega$-axis, which does not depend on $f_\Lambda^2$. However, either $f_\Sigma^2 I$ has to take on a value which is larger than the corresponding quantity for $\pi$-$\mathcal{N}$ scattering [1] or this location is at high energy, $\omega > 2.5$, where the method becomes doubtful.

The only states which can have a resonance in $\pi$-$\Lambda$ as well as $\pi$-$\Sigma$ scattering at low energies are just the remaining possibilities $Y_{(+)}^*$ and $Y_{(-)}^*$ characterized by $J = \frac{1}{2}$ and $T = 1$. At this stage one should examine whether both resonances occur for the same values of the parameters $\omega$, $f_\Lambda^2$ and $f_\Sigma^2 I$ or not. However, from the resonance-equation only necessary conditions can be suitably derived and as is shown in table III the conditions do not demand elimination of any of the two possibilities. Subsequently the special case is considered where the observed $\pi$-$\Lambda$ resonance is identified with $Y_{(-)}^*$ and it is found that in this case $Y_{(+)}^*$ can occur only at high energy: $\omega_+ > 2.5$,

In somewhat more details:

As far as $Y_{(+)}^*$ is concerned it is evident that the requirements for $f_\Lambda^2$ and $f_\Sigma^2 I$ are not independent. Because $f_\Sigma^2$ is analogous to the pion-nucleon coupling constant, it is reasonable to suppose $f_\Sigma^2 I \leqslant 0.1$ which leads to $f_\Lambda^2 > 1.5$. If on the contrary $f_\Sigma^2 I > 0.1$, then for existence of $Y_{(+)}^*$, $f_\Lambda^2$ should be $\leqslant 1.5$, which excludes existence of $Y_{(-)}^*$.

On the other hand, analysis of the resonance-equation for the state $Y_{(-)}^*$ leads to the following results:

The equation can be written in the form:

$$-F(f_\Sigma^2 I, \omega) \cdot L(f_\Lambda^2, \omega) = Q(f_\Lambda^2, \omega) ,$$

where the relevant properties of the functions $F$, $L$ and $Q$ are as follows:

*a*) $F$ is not negative in the intervals $1 < \omega \leqslant 2.4$ (*) and $0 \leqslant f_\Sigma^2 < \infty$.

*b*) $L$ is linear in $f_\Lambda^2$: $L = 1 - f_\Lambda^2 (\omega + \Delta) \varrho_+^+(\omega)$. This function increases as function of $\omega$ and is positive in the $\omega$-interval, mentioned sub *a*) if $f_\Lambda^2 \leqslant 1.5$. For $1.5 < f_\Lambda^2 < 4$ $L$ passes through the $\omega$-axis, the zero is called $\omega(f_\Lambda^2)$.

(*) This is the low energy region for which the numerical calculations have been performed.

*c*) $Q$ is quadratic in $f_\Lambda^2$ and can be written in the form

$$Q = \{1 + f_\Lambda^2 g(\omega)\} L(f_\Lambda^2, \omega) + f_\Lambda^4 h(\omega) ,$$

where $h(\omega) > 0$ and $g(\omega) \gtrsim -0.5$ in the mentioned $\omega$-interval. This function is positive definite in the interval $1.2 < \omega < 2.3$.

From these properties it follows that a resonance is not admitted in the interval $1 < \omega \leqslant 2.4$ if $f_\Lambda^2 \leqslant 1.5$. If $1.5 < f^2 < 4$ the equation can be satisfied for appropriate sets of values of $f_\Lambda^2$, $f_\Sigma^2 I$, $\omega$. A slightly more severe condition for $f_\Lambda^2$ follows from $L < 0$ (in $1.2 < \omega < 2.3$), or $\omega < \omega(f_\Lambda^2)$. For instance at $\omega = 1.9$, which energy corresponds to the observed resonance energy [1], one finds thus $f_\Lambda^2 > 2.5$. It is emphasized here, that these requirements for $f_\Lambda^2$ are independent of the value of $f_\Sigma^2 I$, provided that $f_\Sigma^2 \neq 0$.

In the physically possible limiting case $f_\Sigma^2 = 0$ (we know from the existence of hyperfragments that $f_\Lambda^2$ must be $\neq 0$), all indicated resonances disappear, except $Y_{(-)}^*$. For this state the resonance equation reduces to $Q(f_\Lambda^2, \omega) = 0$. Consequently $\omega$ cannot belong to the interval $1.2 < \omega < 2.3$. This being in contradiction to experimental evidence, one would conclude $f_\Sigma^2 \neq 0$.

As concerns the relative magnitude of $f_\Lambda^2$ and $f_\Sigma^2$, it is noticed that on account of the different nature of scalar and pseudovector interaction, no direct comparison between the two coupling constants can be made. Therefore it would be important to have an independent estimation, for instance for $f_\Lambda^2$, resulting from the study of hyperfragments.

Finally it is mentioned that some support can be given to the speculation that the resonance is actually $Y_{(-)}^*$ which means $P$ —. Decay of $Y_{(-)}^*$ is then possible into $(\pi\Lambda)_s$ and $(\pi\Sigma)_p$, of which the first state is favored, for energy and angular momentum reasons. There is experimental confirmation of such a preference [4]. Taking this for granted, $\omega$ can be taken equal to 1.9, corresponding to the observed energy, which yields « resonating » values of $f_\Lambda^2$ and $f_\Sigma^2 I$. These values are used to calculate $\omega^1$; the result has already been mentioned.

We arrive at the following conclusions:

1) Opposite parities of $\Lambda$ and $\Sigma$ do not allow for a $\pi$-$\Lambda$ resonance in a $T = 1$, $J = \frac{3}{2}$ state, but are consistent with $T = 1$, $J = \frac{1}{2}$ for such a resonance (*).

---

(4) *Conference on Strong Interactions* (Berkeley, 1960).

(*) In ref. (5) an analysis (using the Chew-Low method) is made of both possibilities for the relative $\Lambda$-$\Sigma$ parity. For odd relative parity no $\pi$-$\Lambda$ resonance is found at all, but three $\pi$-$\Sigma$ resonances occur in the same $T$, $J$-state as our $X_i^*$. The differences with our results are due to the very crude approximation of taking the integrals of the type $\varrho(\omega)$ independent of $\omega$ and $\Delta$.

(5) Y. NOGAMI: *Progr. Theor. Phys.*, **22**, 25 (1959).

2) If the observed resonance is actually $Y^*_{(-)}$, than $Y^*_{(+)}$ eventually can occur at an energy $\omega_+ > 2.5$.

* * *

We are indebted to Drs. D. AMATI and J. PRENTKI for their interest in our work and for useful discussions. We want to thank Mr. W. KLEIN for doing numerical calculations.

***Note added in proof.***

Using the resonating values of $f^2_\Lambda$ and $f^2_\Sigma I$, the width of $Y^*_{(-)}$ has been evaluated. In the expression for the width obtained in our model, the parameter $f^2_\Sigma$ occurs separately. Hence these resonating values do not determine uniquely the width. Taking as its value 30 MeV, on the basis of experimental results, both the position and the width of the resonance can be fitted for $f^2_\Lambda \simeq 4$, $f^2_\Sigma I \simeq 0.1$ and $f^2_\Sigma \simeq 0.1$.

## RIASSUNTO

Sulla base dei recenti dati sperimentali riguardanti l'energia e il momento angolare dello stato risonante $\pi$-$\Lambda$ viene eseguita un'analisi dello scattering pione-iperone, con l'ipotesi che $\Lambda$ e $\Sigma$ abbiano parità differenti. Poichè tale risonanza è a bassa energia, si usa una teoria a sorgente fissa. Una giustificazione a tale modello è il fatto che la sua applicazione al caso $\pi$-$\Lambda'$ conduce a buoni risultati. Il risultato è che i dati oggi conosciuti si accordano eglio a dette ipotesi che non a una teoria con simmetria globale.

IL NUOVO CIMENTO Vol. XX, N. 2 16 Aprile 1961

# Physical One-Nucleon and Two-Nucleon Wave Functions in Fixed Source Theory.

J. Mandelbrojt (*)

*CERN - Geneve*

(ricevuto il 20 Febbraio 1961)

**Summary.** — A non perturbative approximation of the physical one-nucleon and two-nucleon wave function in fixed source theory is given. The states are given in an analytic form, from which expectation values of operators can be calculated in an explicit closed form.

## Introduction.

In this paper we give a non perturbative calculation of the physical one-nucleon and two-nucleon wave function in fixed source theory. Our calculation introduces in a very natural manner the creation operators of mesons in a state of average momentum, which form the basis of the usual intermediate coupling theory. The other advantage of our technique over the usual intermediate coupling technique lies in the simplicity of the calculations and in the fact that we get explicit workable analytic expressions for the physical one-nucleon and two-nucleon states, thus avoiding the use of machines and giving more transparent results. Furthermore the approximation with which we approach the exact fixed source wave function can be evaluated in a straightforward manner. It is to be noted that the method we use can, for a number of problems, replace variational calculations.

We intend eventually to use the two-nucleon wave function that we obtain to carry out the program outlined by Lévy [1], which comprises two main points:

(*) Present address: Physique théorique, Faculté des Sciences, Place Victor-Hugo, Marseille, France

1) to obtain in a straightforward way a nucleon-nucleon potential which comprises terms of all orders of perturbation and should therefore improve in particular the part not due to $\pi$-$\pi$ interaction, of the spin orbit potential which begins to appear in forth order perturbation;

2) to calculate in detail the imaginary potential in the interaction nucleon-antinucleon, Lévy's calculation being so far the only theoretical justification for a potential having the range which is implied by experimental results, that is $1.5 \cdot 10^{-13}$ cm. (*).

## 1. – Fixed-source nucleon surrounded by $P$-wave mesons.

The hamiltonian representing $P$-wave mesons interacting with a nucleon without recoil can be written as

$$H = \frac{1}{(2\pi)^3} \int \left[ \omega(k) a^*_{i\alpha}(\boldsymbol{k}) a_{i\alpha}(\boldsymbol{k}) - f R(k) \sigma_i \tau_\alpha \left( a^*_{i\alpha}(\boldsymbol{k}) + a_{i\alpha}(\boldsymbol{k}) \right) \right] \mathrm{d}\boldsymbol{k} , \tag{1}$$

where $a^*_{i\alpha}(\boldsymbol{k})$ and $a_{i\alpha}(\boldsymbol{k})$ are the creation and annihilation operators of mesons of spin and isotopic spin indices $i$ and $\alpha$ and momentum vector $\boldsymbol{k}$, $\omega(k)$ its total energy, $R(k)$ is the spherically symmetrical source function.

We can write $H$ in a form which is similar to the canonical form for a second degree polynomial:

$$H = \frac{1}{(2\pi)^3} \int \left\{ \omega(k) \left( a^*_{i\alpha}(\boldsymbol{k}) - \frac{f R(k)}{\omega(k)} \sigma_i \tau_\alpha \right) \left( a_{i\alpha}(\boldsymbol{k}) - \frac{f R(k)}{\omega(k)} \sigma_i \tau_\alpha \right) - 9 \frac{f^2 R^2(\omega)}{\omega(k)} \right\} \mathrm{d}\boldsymbol{k} .$$

In this form we see that the ground state $|\psi\rangle$ of $H$ will be obtained by minimizing

$$\left\langle \psi^* \left( a^*_{i\alpha}(\boldsymbol{k}) - \frac{f R(k)}{\omega(k)} \sigma_i \tau_\alpha \right) \left( a_{i\alpha}(\boldsymbol{k}) - \frac{f R(k)}{\omega(k)} \sigma_i \tau_\alpha \right) \psi \right\rangle .$$

It is clear that if the system of equations

$$\left( a_{i\alpha}(\boldsymbol{k}) - \frac{f R(k)}{\omega(k)} \sigma_i \tau_\alpha \right) |\psi\rangle = 0 \tag{2}$$

is compatible, its solution will be the ground state and the corresponding energy

(*) Range given in the 1960 Rochester Report on nucleon-antinucleon forces by M. Lévy.

will be

$$-\frac{1}{(2\pi)^3}\int 9\,\frac{f^2R^2(k)}{\omega(k)}\,\mathrm{d}^3k\,.$$

This would be the case in the neutral scalar model where, when the indices $i$, $\alpha$ and the matrices $\sigma_i$ and $\tau_\alpha$ are suppressed, the equations

$$\left(a(\boldsymbol{k})-f\,\frac{R(k)}{\omega(k)}\right)\psi\rangle=0\,,$$

which correspond to the system (2) are compatible and yield

$$\psi\rangle=B\exp\left[A\int\frac{fR(k)}{\omega(k)}\;a^*(\boldsymbol{k})\,\mathrm{d}\boldsymbol{k}\right]\psi_0\rangle\,,$$

where $\psi_0\rangle$ is the bare nucleon and $A$ and $B$ are normalization constants defined by

$$A^2\int\frac{f^2R^2(h)}{\omega^2(k)}\,\mathrm{d}\boldsymbol{k}=1\qquad\text{and}\qquad\langle\psi^*\psi\rangle=1\,.$$

We have studied the system (2) and have shown that it is incompatible as a result of the commutation relations of the matrices $\sigma$, and $\tau$. We have given a quantitative meaning to the incompatibility of these equations thus being able to assert to what approximation these equations can be considered as compatible. From eq. (2) we can obtain the two systems of equations

$$a^*_{i\alpha}(\boldsymbol{k})a_{i\alpha}(\boldsymbol{k})\psi\rangle=f\,\frac{R(k)}{\omega(k)}\,\sigma_i\tau_\alpha a^*_{i\alpha}(\boldsymbol{k})\,\psi\rangle\,,\tag{3}$$

and

$$\sigma_i\tau_\alpha a_{i\alpha}(\boldsymbol{k})\psi\rangle=g(k)\psi\rangle\,,\tag{4}$$

$g(k)$ being a $c$ function. It is easily checked that if these two systems of equations are satisfied simultaneously, $\psi\rangle$ will be an eigenstate of $H$. These two systems are in fact satisfied together in the weak and in the strong coupling limit [1]. In the weak coupling limit $g(k)=9\big(fR(k)/\omega(k)\big)$ as would be expected from eq. (2), in the strong coupling limit $g(k)=\big(fR(k)/\omega(k)\big)$. The fact that the systems (3) and (4) are compatible in the weak and in the strong coupling limit expresses, as we have shown in reference [1], the fact that Tomo-

[1] J. MANDELBROJT: *Nuovo Cimento*, **14**, 625 (1959).

naga's intermediate coupling theory is valid in the weak and in the strong coupling limit.

It should be noted that the equation

$$\langle \psi^* \left( a_{i\alpha}(\boldsymbol{k}) - f \frac{R(k)}{\omega(k)} \sigma_i \tau_\alpha \right) \psi \rangle = 0 ,$$

is rigorously satisfied in the ground state as Fubini has shown ([2]). The arguments Fubini gives at page 294 of reference ([2]) show in fact that the system of eq. (2) can be considered as valid with a good approximation when one is interested in the scattering state of one meson scattered by the clothed nucleon.

*Solution of the system of eq.* (3). – The solution of the system (3) is immediately obtained; it is

$$\psi\rangle = B \exp \left[ \int \frac{fR(k)}{\omega(k)} \sigma_i \tau_\alpha a^*_{i\alpha}(\boldsymbol{k}) \, \mathrm{d}\boldsymbol{k} \right] \psi_0\rangle ,$$

which we can write in a more condensed form by introducing the creation operator

$$a^*_{i\alpha} = \frac{1}{(2\pi)^2} \int a^*_{i\alpha}(\boldsymbol{k}) \frac{Af R(k)}{\omega(k)} \, \mathrm{d}\boldsymbol{k} ,$$

$$\psi\rangle = B \exp \left[ \frac{f_0}{\omega_0} \sigma_i \tau_\alpha a^*_{i\alpha} \right] \psi_0\rangle ,$$

where $A$ which we shall call $\omega_0/f_0$ and $B$ are normalization constants defined by

$$\frac{A^2}{(2\pi)^3} \int \frac{f^2 R^2(k)}{\omega^2(k)} \, \mathrm{d}\boldsymbol{k} = 1 \qquad \text{and} \qquad \langle \psi^* \psi \rangle = 1 .$$

We see that operators of the form $a^*_{i\alpha}(\Phi) = 1/(2\pi)^3 \int a^*_{i\alpha}(\boldsymbol{k}) \, \Phi(\boldsymbol{k}) \, \mathrm{d}\boldsymbol{k}$ (with $1/(2\pi)^3 \int \Phi^2(\boldsymbol{k}) \, \mathrm{d}\boldsymbol{k} = 1$) which form the basis of Tomonaga's intermediate coupling theory, are thus introduced in a natural manner.

For practical calculations of mean values of operators this form is not convenient because of the complications of the commutation relations of high powers of $\sigma_i \tau_\alpha$. We therefore give a reduced form to $\psi\rangle$ following a method given by Friedman, Lee and Christian ([3]).

The ground state of the nucleon being like the bare nucleon of spin and isotopic spin $\frac{1}{2}$, it is obtained from the bare nucleon by application of an

([2]) S. Fubini: *Suppl. Nuovo Cimento*, **14**, 283 (1959).

([3]) M. H. Friedman, T. D. Lee and R. Christian: *Phys. Rev.*, **100**, 1494 (1955).

operator constructed with the operators $\sigma_i$, $\tau_\alpha$ and $a^*_{i\alpha}$ which must be a scalar in space and in isospace. F.L.C. have proved that the most general form of such an operator is

$$f_0(s_1, s_2, s_3) + f_1(s_1, s_2, s_3)\sigma_i\tau_\alpha a^*_{i\alpha} + f_2(s_1, s_2, s_3)\sigma_i\tau_\alpha \varepsilon_{ijk}\varepsilon_{\alpha\beta\gamma} a^*_{j\beta} a^*_{k\gamma} + \\ + f_3(s_1, s_2, s_3)\sigma_i\tau_\alpha a^*_{j\beta} a^*_{i\beta} a^*_{j\alpha},$$

where

$$s_1 = a^*_{i\alpha} a^*_{i\alpha}$$

$$s_2 = \varepsilon_{ijk}\varepsilon_{\alpha\beta\gamma} a^*_{i\alpha} a^*_{j\beta} a^*_{k\gamma}$$

$$s_3 = a^*_{i\alpha} a^*_{j\beta} a^*_{i\beta} a^*_{j\alpha}$$

$f_0$, $f_1$, $f_2$, $f_3$ being scalar functions.

Furthermore it has been noticed ([3]) that variables in the $a^*_{i\alpha}$ space can be introduced for which the wave function $\psi\rangle$ has cubic symmetry, and approximating this cubic symmetry by a spherical symmetry is equivalent to considering that $f_0$, $f_1$, $f_2$, $f_3$ are functions of the only variable $s_1$.

To represent $\psi\rangle = B \exp[(f_0/\omega_0)\sigma_i\tau_\alpha a^*_{i\alpha}]\psi_0\rangle$ in the reduced form

$$\psi_R\rangle = B\{f_0(s_1) + f_1(s_1)\sigma_i\tau_\alpha a^*_{i\alpha} + f_2(s_1)\sigma_i\tau_\alpha \varepsilon_{ijk}\varepsilon_{\alpha\beta\gamma} a^*_{j\beta} a^*_{k\gamma} + f_3(s_1)\sigma_i\tau_\alpha a^*_{j\beta} a^*_{i\beta} a^*_{j\alpha}\}\psi_0\rangle,$$

where we will write:

$$f_0(s_1) = \sum u_n s_1^n, \qquad f_1(s_1) = \sum v_n s_1^n, \qquad f_2(s_1) = \sum w_n s_1^n, \qquad f_3(s_1) = \sum c_n s_1^n$$

or, what is equivalent, to look for the best approximation of the form (6) to a solution of $a^*_{i\alpha} a_{i\alpha}\psi\rangle = (f_0/\omega_0)\sigma_i\tau_\alpha a^*_{i\alpha}\psi\rangle$, we determine the coefficients $u_n$, $v_n$, $w_n$, $c_n$ by the condition that

$$\left(a^*_{i\alpha}a_{i\alpha} - \frac{f_0}{\omega_0}\sigma_i\tau_\alpha a^*_{i\alpha}\right)(f_0(s_1) + f_1(s_1)\sigma_i\tau_\alpha a^*_{i\alpha} + f_2(s_1)\sigma_i\tau_\alpha \varepsilon_{ijk}\varepsilon_{\alpha\beta\gamma} a^*_{j\beta} a^*_{k\gamma} + \\ + f_3(s_1)\sigma_i\tau_\alpha a^*_{j\beta} a^*_{i\beta} a^*_{j\alpha})\psi_0\rangle,$$

is minimum. Using the value given in Appendix I of $\sigma_i\tau_\alpha a^*_{i\alpha}$ applied to each of the terms of $\psi_R\rangle$ and the obvious value of $a^*_{i\alpha}a_{i\alpha}$ applied to each of the terms of $\psi_R\rangle$ we get

$$u_n = \frac{3^n (f_0/\omega_0)^{2n-1}}{2n!}$$

$$v_n = \frac{3^{n+1}(f_0/\omega_0)^{2n}}{(2n+1)!},$$

$$w_n = \frac{-3^{n+1}(f_0/\omega_0)^{2n+1}}{(2n+2)!},$$

$$c_n = \frac{-2\cdot 3^{n+1}(f_0/\omega_0)^{2n+2}}{(2n+3)!}.$$

The ground state expectation value of an operator can now be obtained explicitly using the relations given in Appendix III and the values of $u_n$, $v_n$, $w_n$, $c_n$. It is essential to note that $\langle\psi_R^*\psi_R\rangle$, $\langle\psi_R^* a_{i\alpha}^* a_{i\alpha}\psi_R\rangle$ and $\langle\psi_R^*\sigma_i\tau_\alpha a_{i\alpha}\psi_R\rangle$ can by this calculation be given in analytically closed form, for instance

$$\langle\psi_R^*\psi_R\rangle = \frac{2B^2}{5\cdot 6\cdot 7}\left[\cosh z(z^4+26z^3+141z^2+561z-105)\right. +$$
$$\left. + \sinh z(z^4+22z^3+207z^2+279z+384)\right],$$

$$\langle\psi_R^*\sigma_i\tau_\alpha a_{i\alpha}\psi_R\rangle = \frac{2B^2}{5\cdot 6\cdot 7}\frac{f_0}{w_0}\left[\sinh z^3(67z^3+975z)+\cosh z(z^4+285z^2+945)\right] +$$
$$+\frac{2B}{5\cdot 6\cdot 7}\frac{1}{f^0/\omega_0}\frac{\left[\sinh z(z^4+207z^2+384)+\cosh z(26z^3+561z)\right]}{z^3},$$

with $z=(f_0/\omega_0)^2$.

## 2. – Two-nucleon wave function.

We shall relate the calculation of the two-nucleon wave function to that of the clothed-nucleon wave function which we calculated in the preceding paragraph.

The fixed source hamiltonian of the nucleon-nucleon system interacting via $P$ wave mesons is

$$H = \frac{1}{(2\pi)^3}\int\left\{\omega(k)a_\alpha^*(\boldsymbol{k})a_\alpha(\boldsymbol{k}) + \frac{f}{\mu}\sum_{j=1}^{2}R(k)\tau_\alpha^{(j)}\left[i(\boldsymbol{\sigma}^{(j)}\cdot\boldsymbol{k})\,a_\alpha(\boldsymbol{k})\exp\left[i\boldsymbol{k}\boldsymbol{x}_j\right]-\right.\right.$$
$$\left.\left. - i(\sigma^{(j)}\cdot\boldsymbol{k})a_\alpha^*(\boldsymbol{k})\exp\left[-i\boldsymbol{k}\boldsymbol{x}_j\right]\right]\right\}\mathrm{d}^3k,$$

where $a_\alpha^*(k)$ and $a_\alpha^*(k)$ are the creation and annihilation operators of the meson field, $\alpha$ is the charge indice. The terms $\exp[ikx_j]$ and $\exp[-ikx_j]$ come from the fact that we have two sources, for the meson field, the two nucleons whose co-ordinates are $x_1$ and $x_2$, we take $x_1=-x_2=r/2$.

Again we write $H$ in the form

$$H = \frac{1}{(2\pi)^3}\int \omega(k)\left\{a_\alpha^*(\boldsymbol{k}) - i\frac{f}{\mu}\sum_{j=1}^{2}\frac{R(k)}{\omega(k)}\tau_\alpha^{(j)}(\boldsymbol{\sigma}^{(j)}\cdot\boldsymbol{k})\exp[i\boldsymbol{k}\boldsymbol{x}_j]\right\}\cdot$$

$$\cdot\left\{a_\alpha(\boldsymbol{k}) - i\frac{f}{\mu}\sum_{j=1}^{2}\frac{R(k)}{\omega(k)}\tau_\alpha^{(j')}(\boldsymbol{\sigma}^{(j')}\cdot\boldsymbol{k})\exp[-i\boldsymbol{k}\boldsymbol{x}_{j'}]\right\}\mathrm{d}^3k -$$

$$-\frac{1}{(2\pi)^3}\frac{f^2}{\mu^2}\int\frac{R^2(k)}{\omega(k)}\sum_{j,j'=1}^{2}\tau_\alpha^{(j)}\tau_\alpha^{(j')}(\boldsymbol{\sigma}^{(j)}\cdot\boldsymbol{k})(\sigma^{(j')}\cdot\boldsymbol{k})\exp\left[i\boldsymbol{k}(\boldsymbol{x}_j-\boldsymbol{x}_{j'})\right]\mathrm{d}\boldsymbol{k}\,.$$

As in the first paragraph we are led to the study of the compatibility of the system

$$a_\alpha(\boldsymbol{k})\psi\rangle = i\frac{f}{\mu}\sum_{j=1}^{2}\frac{R(k)}{\omega(k)}\tau_\alpha^{(j)}(\boldsymbol{\sigma}^{(j)}\cdot\boldsymbol{k})\exp[-i\boldsymbol{k}\boldsymbol{x}_j]\psi\rangle\,,$$

which corresponds to eq. (2). If this system were compatible it would lead to a nucleon-nucleon potential of the form

$$V(r) = \left[\frac{1}{(2\pi)^3}\frac{f^2}{\mu^2}\int\frac{R^2(k)}{\omega(k)}\sum_{j,j'=1}^{2}\tau_\alpha^{(j)}\tau_\alpha^{(j')}(\boldsymbol{\sigma}^{(j)}\cdot\boldsymbol{k})(\boldsymbol{\sigma}^{(j')}\cdot\boldsymbol{k})\exp\left[i\boldsymbol{k}(\boldsymbol{x}_j-\boldsymbol{k}_{j'})\right]\mathrm{d}\boldsymbol{k}\right]_r^\infty,$$

$[\ ]_r^\infty$ meaning the difference of the quantity for $x_j - x_{j'} = r$ and $x_j - x_{j'} = \infty$ (which is twice the self-energy of the nucleon) (*). The equation corresponding to eq. (3) of the first paragraph will yield

$$\psi\rangle = B\exp\left[i\frac{f}{\mu}\int\sum_{j=1}^{2}\frac{R(k)}{\omega(k)}\tau_\alpha^{(j)}(\boldsymbol{\sigma}^{(j)}\cdot\boldsymbol{k})\exp[-i\boldsymbol{k}\cdot\boldsymbol{x}_j]\,\mathrm{d}\boldsymbol{k}\right]\psi_{00}\rangle\,,$$

being the state vector of the system of the two bare nucleons, this can be written as

$$\psi\rangle = B^+B^-\exp\left[\sqrt{2}\,\frac{f^+}{\omega_+(r)}(\sigma_i^1\tau_\alpha^1+\sigma_i^2\tau_\alpha^2)a_{i\alpha}^{+*}\right]\exp\left[i\sqrt{2}\,\frac{f^-}{\omega_-(r)}(\sigma_i^1\tau_\alpha^1-\sigma_i^2\tau_\alpha^2)a_{i\alpha}^{-*}\right]\psi_{00}\rangle\,,$$

where

$$a_{i\alpha}^{+*} = \frac{i\sqrt{6}}{(2\pi)^3}\int A^+\frac{fR(k)}{\omega(k)}\,\frac{k_i}{k}\cos\frac{\boldsymbol{k}\cdot\boldsymbol{r}}{2}\,a^*(\boldsymbol{k})\,\mathrm{d}\boldsymbol{k}\,,$$

(*) This potential is identical to that given by the first Born approximation.

and

$$a_{i\lambda}^{-*} = \frac{i\sqrt{6}}{(2\pi)^3}\int A^- \frac{fR(k)}{\omega(k)}\frac{k_i}{k}\sin\frac{\boldsymbol{k}\cdot\boldsymbol{r}}{2}\,a_\lambda^*(\boldsymbol{k})\,\mathrm{d}\boldsymbol{k}\,,$$

$f^-/(\omega_-(r))$ and $f^+/(\omega_+(r))$ are the normalization constants $1/A^-$ and $1/A^+$ defined as

$$\frac{1}{(2\pi)^3}\int (A^\pm)^2\frac{f^2R^2(k)}{\omega^2(k)}\,(1\pm\cos\boldsymbol{k}\cdot\boldsymbol{r})\,\mathrm{d}\boldsymbol{k} = 1\,.$$

We again can write this function in the reduced form by simply taking the product of the reduced forms of $\psi^+\rangle$ and $\psi^-\rangle$:

$$\psi^+\rangle = B^+\exp\left[\sqrt{2}\,\frac{f^+}{\omega+(r)}\,(\sigma_i^1\tau_\alpha^1+\sigma_i^2\tau_\alpha^2)a_{i\alpha}^{+*}\right]\psi_{00}\rangle\,,$$

$$\psi_R^+ = B^+\Big(\sum_n u_n(a_{i\alpha}^{+*2})^n + v_n(a_{i\alpha}^{+*2})^n\sigma_i^1\tau_\alpha^1 a_{i\alpha}^{+*} + w_n(a_{\alpha i}^{+*2})^n\sigma_i^1\tau_\alpha^1\varepsilon_{ijk}\varepsilon_{\alpha\beta\gamma}a_{j\beta}^{+*}a_{k\gamma}^* +$$
$$+ c_n(a_{i\alpha}^{+*2})^n\sigma_i\tau_\alpha^1 a_{j\beta}^{+*}a_{i\beta}^{+*}a_{j\alpha}^{*+}\Big)\Big(\sum_n u_n(a_{i\alpha}^{+*2})^n + v_n(a_{i\alpha}^{+*2})^n\sigma_i^2\tau_\alpha^2 a_{i\alpha}^{+*} +$$
$$+ w_n(a_{i\alpha}^{+*2})^n\sigma_i^2\tau_\alpha^2\varepsilon_{ijk}\varepsilon_{\alpha\beta\gamma}a_{j\beta}^{+*}a_{k\gamma}^{+*} + c_n(a_{i\alpha}^{+*2})^n\sigma_i^2\tau_\alpha^2 a_{j\beta}^{+*}a_{i\beta}^{+*}a_{j\alpha}^{+*}\Big)\psi_{00}\rangle\,,$$

where $u_n$, $v_n$, $w_n$, $c_n$ have the values previously calculated where $f_0/\omega_0$ is replaced by $\sqrt{2}\,f^+/(\omega_+(r))$, $\psi_R^-\rangle$ will have a similar expression with $f_0/\omega_0$ replaced by $(-i\sqrt{2}f_-)/(\omega_-(r))$.

Calculations using these expressions to carry out the program outlined in the introduction are under way.

* * *

We wish to thank Professor LÉVY for his interest in this work and Dr. ANDRÉ MARTIN for enjoyable discussions. We are also grateful to Professor FUBINI for having read this paper and for useful criticism.

## APPENDIX I

$$\sigma_i\tau_\alpha a_{i\lambda}^*(a_{i\alpha}^{*2})^n = \sigma_i\tau_\alpha a_{i\alpha}^*(a_{i\alpha}^{*2})^n\,,$$

$$\sigma_j\tau_\beta a_{j\beta}^*\sigma_i\tau_\alpha a_{i\alpha}^*(a_{i\alpha}^{*2})^n = (a_{i\alpha}^{*2})^{n+1} - \sigma_k\tau_\gamma\varepsilon_{ijk}\varepsilon_{\alpha\beta\gamma}a_{i\alpha}^*a_{j\beta}^*(a_{i'\alpha'}^{*2})^n\,,$$

$$\sigma_{i'}\tau_{\alpha'}a_{i'\alpha'}^*\sigma_i\tau_\alpha\varepsilon_{ijk}\varepsilon_{\alpha\beta\gamma}a_{j\beta}^*a_{k\gamma}^*(a_{i''j''}^{*2})^n = a_{i\alpha}^*a_{j\beta}^*a_{k\gamma}^*\varepsilon_{ijk}\varepsilon_{\alpha\beta\gamma}(a_{i''\alpha''}^{*2})^n -$$
$$- 2\sigma_k\tau_\gamma a_{k\gamma}^*(a_{i''\alpha''}^{*2})^{n+1} + 2\sigma_k\tau_\beta a_{j\beta}^*a_{k\gamma}^*a_{j\gamma}^*(a_{i''\alpha''}^{*2})^n\,.$$

$$\sigma_{i'}\tau_{\alpha'}a_{i'\alpha'}^*\sigma_i\tau_\alpha a_{j\beta}^*a_{i\beta}^*a_{j\alpha}^*(a_{i''\alpha''}^{*2})^n = a_{i\alpha}^*a_{j\beta}^*a_{i\beta}^*a_{j\alpha}^*(a_{i''\alpha''}^{*2})^n - \sigma_{k'}\tau_{\gamma'}\varepsilon_{i'ik'}\varepsilon_{\alpha'\alpha\gamma'}a_{i'\alpha'}^*a_{j\beta}^*a_{i\beta}^*a_{j\alpha}^*(a_{i''\alpha''}^{*2})^n\,.$$

(4) M. LÉVY: *Nuovo Cimento*, **8**, 92 (1958).

## Appendix II

$$\sigma_i\tau_\alpha a_{i\alpha}(a^{*2}_{i'\alpha'})^n\psi_0\rangle = 2n\sigma_i\tau_\alpha(a^{*2}_{i'\alpha'})^{n-1}\psi_0\rangle\,,$$

$$\sigma_i\tau_\alpha a_{i\alpha}\sigma_{i'}\tau_{\alpha'}a_{i'\alpha'}(a^{*2}_{i''\alpha''})^n\psi_0\rangle = \left\{(9+2n)(a^{*2}_{i\alpha})^n - 2n\sigma_k\tau_\beta\varepsilon_{\alpha\alpha'\beta}\varepsilon_{ii'k}a^*_{i\alpha}a^*_{i'\alpha'}(a^{*2}_{i''\alpha''})^n\right\}\psi_0\rangle\,,$$

$$\sigma_{i'}\tau_{\alpha'}a_{i'\alpha'}\sigma_i\tau_\alpha\varepsilon_{ijk}\varepsilon_{\alpha\beta\gamma}a^*_{j\beta}a^*_{k\gamma}(a^{*2}_{i''\alpha''})^n\psi_0\rangle = \left\{2n\varepsilon_{ijk}\varepsilon_{\alpha\beta\gamma}a^*_{j\beta}a^*_{k\gamma}a^*_{i\alpha}(a^{*2}_{i''\alpha''})^{n-1} + \right.$$
$$\left. + 4n\sigma_k\tau_\beta a^*_{j\beta}a^*_{k\beta}a^*_{j\gamma}(a^{*2}_{i''\alpha''})^{n-1} - (4n+8)\sigma_k\tau_\gamma a^*_{k\gamma}(a^{*2}_{i''\alpha''})^n\right\}\psi_0\rangle\,,$$

$$\sigma_{i'}\tau_{\alpha'}a_{i'\alpha'}\sigma_i\tau_\alpha a^*_{j\beta}a^*_{i\beta}a^*_{j\alpha}(a^{*2}_{i''\alpha''})^n\psi_0\rangle = \left\{2na^*_{j\beta}a^*_{i\beta}a^*_{j\alpha}a^*_{i\alpha}(a^{*2}_{i''\alpha''})^{n-1} - \right.$$
$$- 2n\sigma_{k'}\tau_{\beta'}\varepsilon_{i'ik'}\varepsilon_{\alpha'\alpha\beta'}a^*_{j\beta}a^*_{i\beta}a^*_{j\alpha}a^*_{i'\alpha'}(a^{*2}_{i''\alpha''})^{n-1} + 7(a^{*2}_{j''\alpha''})^{n+1} -$$
$$\left. - \sigma_{k'}\tau_{\beta'}\varepsilon_{i'ik'}\varepsilon_{\alpha'\alpha\beta'}a^*_{i\alpha'}a^*_{i'\alpha}(a^{*2}_{i''\alpha''})^n\right\}\psi_0\rangle\,.$$

## Appendix III

The following expectation values obtained by comparison of the expressions given in Appendix I and II

$$A_n = \langle(a^2_{i\alpha})^n(a^{*2}_{i'\alpha'})^n\rangle = \frac{(2n+7)!}{4\cdot5\cdot6\cdot7(n+1)(n+2)(n+3)}\,,$$

$$B_n = \langle(\ )^n\sigma_{i'}\tau_{\alpha'}a_{i'\alpha'}\sigma_i\tau_\alpha a^*_{i\alpha}(\ )_n\rangle = (2n+9)A_n\,,$$

where $(\ )^n\rangle$ stands for $(a^{*2}_{i\alpha})^n\rangle$ and $\langle(\ )^n$ stands for $\langle(a^2_{i\alpha})^n$

$$C_n = \langle(\ )^{n+1}\sigma_{i'}\tau_{\alpha'}a_{i'\alpha'}\sigma_i\tau_\alpha a_{j\beta}a^*_{i\beta}a^*_{j\alpha}(\ )^n\rangle = 4_n(n+1)C_{n-1} + 7A_{n+1}\,,$$

$$D_n = \langle(\ )^n\sigma_{i''}\tau_{\alpha''}a_{i''\alpha''}\sigma_{i'}\tau_{\alpha'}a_{i'\alpha'}\sigma_i\tau_\alpha\varepsilon_{ijk}\varepsilon_{\alpha\beta\gamma}a^*_{j\beta}a^*_{k\gamma}(\ )^n\rangle = 4nC_{n-1} - (4n+8)B_n\,,$$

$$E_n = \langle(\ )^n\sigma_{i''}\tau_{\alpha''}\varepsilon_{i''j''k''}\varepsilon_{\alpha''\beta''\gamma''}a_{j''\beta''}a_{k''\gamma''}\sigma_{i'}\tau_{\alpha'}a_{i'\alpha'}\sigma_i\tau_\alpha a^*_{j\beta}a^*_{i\beta}a^*_{j\alpha}(\ )^n\rangle = 4n^2E_{n-1} - 16C_{n-1} + D_n\,,$$

$$F_n = \langle(\ )^{n-1}\sigma_{i''}\tau_{\alpha''}a_{j''\alpha''}a_{i''\beta''}a_{j''\beta''}\sigma_{i'}\tau_{\alpha'}a_{i'\alpha'}\sigma_i\tau_\alpha\varepsilon_{ijk}\varepsilon_{\alpha\beta\gamma}a^*_{j\beta}a^*_{k\gamma}(\ )^n\rangle = \frac{1}{2n}E_n - \frac{1}{2n}D_n\,,$$

$$H_n = \langle(\ )^n\sigma_{i''}\tau_{\alpha''}\varepsilon_{i''j''k''}\varepsilon_{\alpha''\beta''\gamma''}a_{j''\beta''}a_{k''\gamma''}\sigma_i\tau_\alpha a_{i\alpha}\sigma_{i'}\tau_{\alpha'}a^*_{i'\alpha'}(\ )^{n+1}\rangle = 2(n+1)D_n\,,$$

$$L_n = \langle(\ )^n\sigma_i\tau_\alpha a_{j\beta}a_{i\beta}a_{j\alpha}\sigma_{i'}\tau_{\alpha'}a_{i'\alpha'}(\ )^{n+2}\rangle = \frac{1}{2(n+1)}C_{n+1} - \frac{7}{2(n+1)}A_{n+2}\,,$$

$$M_n = \langle(\ )^n\sigma_i\tau_\alpha\varepsilon_{ijk}\varepsilon_{\alpha\beta\gamma}a_{j\beta}a_{k\gamma}\sigma_{i'}\tau_{\alpha'}\varepsilon_{i'j'k'}\varepsilon_{\alpha'\beta'\gamma'}a^*_{j'\beta'}a^*_{k'\gamma'}(\ )^n\rangle = -D_n\,,$$

$$N_n = \langle(\ )^n\sigma_{i'}\tau_{\alpha'}a_{j'\beta'}a_{i'\beta'}a_{j'\alpha'}\sigma_i\tau_\alpha a^*_{j\beta}a^*_{i\beta}a^*_{j\alpha}(\ )^n\rangle = \tfrac{1}{2}E_n + C_n\,.$$

---


## RIASSUNTO (*)

Si dà una approssimazione non perturbativa della funzione d'onda fisica a nucleone singolo e a due nucleoni nella teoria della sorgente fissa. Si danno gli stati in forma analitica, da cui si possono calcolare i valori di aspettazione degli operatori in forma esplicita chiusa.


(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. XX, N. 2 16 Aprile 1961

# A Model for Double Photoproduction of Charged Pions – II.

D. Boccaletti (*), G. De Franceschi and C. Gualdi (*)

*Scuola di Perfezionamento in Fisica Nucleare - Roma*

(ricevuto il 24 Febbraio 1961)

**Summary.** — Using the matrix element for double photoproduction of charged pions constructed in the first part of this paper, the $\pi^-$ energy spectrum at three fixed angles and the $\pi^-$ angular distribution at three fixed energies are calculated. Also the $\frac{3}{2}\frac{3}{2}$ final state interaction between $\pi^+$ and proton has been taken into account: the resulting correction changes very little the form of the $\pi^-$ spectrum. The aim of our calculation is to see if it is possible to separate experimentally the predictions made by our diagram from the Drell's ones, being zero the interference term of the two graphs. As in the first part, the $\gamma$ energy is taken of 900 MeV.

## 1. – Introduction.

The reaction

$$\gamma + p \to p + \pi^+ + \pi^- \tag{1}$$

has been studied in the first part of this paper (**) on the basis of the diagram of Fig. 1.

Taking into account that, at small proton momentum transfers, the matrix element written in I is dominated by the propagator $(\Delta^2+\mu^2)^{-1}$, it has been

(*) Permanent address: Istituto di Fisica dell'Università, Bologna. This work has been done whilst the authors were at the Università di Roma with a scholarship of the Istituto Nazionale di Fisica Nucleare.

(**) D. Boccaletti and C. Gualdi: *Nuovo Cimento*, **18**, 895 (1960); hereafter it will be referred to as I.

assumed that the diagram of Fig. 1 gives a good approximation to the problem at least in these kinematical conditions. Actually, using similar arguments on the location of the poles of the scattering matrix, one can see that in the same kinematical conditions (*i.e.* small recoil energy of the proton) also the Drell's (*) diagram of Fig. 2 must be considered.

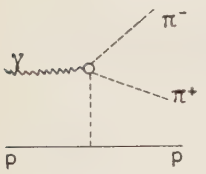

Fig. 1.

In fact the Drell's matrix element is dominated by the propagator $(d^2+\mu^2)^{-1}$, where $d^2=(K-q_2)^2$ is the square of the $\pi^-$ momentum transfer.

In the laboratory system:

$$d^2+\mu^2=2K_{0L}q_{20L}\left(1-\frac{|\boldsymbol{q}_{2L}|}{q_{20L}}\cos\alpha_L\right).$$

Therefore at small angles and at high $\pi^-$-energies $((|\boldsymbol{q}_{2L}|/q_{20L})\sim 1)$ the propagator $(d^2+\mu^2)^{-1}$ becomes very large. Moreover the smallness of the electromagnetic coupling constant is compensated by the resonant behaviour of the lower vertex where the $\pi^+$-p cross-section appears. Therefore at high $\pi^-$ energies the contribution of the diagram of Fig. 2 is certainly important.

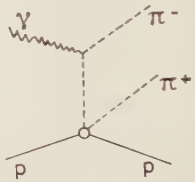

Fig. 2.

Now to high energies of the $\pi^-$ correspond small proton energies and so the two graphs are important in the same kinematical region. When one has to calculate total cross-sections things may change. In fact from the Drell's diagram one obtains a total cross-section of about 9 μb for $E_\gamma=900$ MeV. The experimental one at this energy is about 60 μb. It is clear that the Drell's diagram contribution is important only in the above mentioned kinematical region.

Therefore we shall calculate in this paper both the $\pi^-$ energy spectrum and its angular distribution by means of the diagram of Fig. 1. The aim is to see if predictions different from the Drell's ones can be made. In this case, being zero the interference term of the two graphs (**), an experimental check of the two graphs will be greatly simplified.

With the aim of obtaining the best possible evaluation of the effects given by our diagram, we have also calculated the corrections to the $\pi^-$ energy spectrum due to the $\pi^+$-p final state interaction. In the following calculations a $\gamma$ energy of 900 MeV is assumed.

---

(*) S. D. DRELL: *Phys. Rev. Lett.*, **5**, 278 (1960).

(**) We are indebted for this remark to prof. A. STANGHELLINI.

## 2. – Final state interaction.

The cross-section for the reaction (1) is written in terms of the $R$ matrix as

$$(2) \qquad d\sigma = (2\pi)^2 \frac{K_0 P_{10}}{(KP_1)} \int_f \int_i \delta^4(q_2 + q_3 + P_2 - K - P_1) \, |\langle q_2 q_3 P_2 | M | K P_1 \rangle|^2 ,$$

where

$$R = \delta^4(P_f - P_i) M \quad (*)$$

having used as in I the following notations:

| | | |
|---|---|---|
| $K$ | 4-momentum | of the incident photon |
| $P_1$ | » | of the initial proton |
| $q_1$ | » | of the intermediate pion |
| $q_2$ | » | of the final $\pi^-$ |
| $q_3$ | » | of the final $\pi^+$ |
| $P_2$ | » | of the final proton . |

We also define the following invariants which will be useful later on:

$$W^2 = -(K + P_1)^2 \qquad u^2 = -(q^3 + P_2)^2$$
$$\Delta^2 = (P_2 - P_1)^2 \qquad r^2 = (q_2 - P_1)^2$$
$$w^2 = -(q_2 + q_3)^2 \qquad d^2 = (K - q_2)^2 .$$

Being five the independent invariants of our problem, between the six ones before defined there is the relation

$$d^2 + r^2 + u^2 = W^2 - m^2 - \mu^2 .$$

Now we want to calculate (2) taking into account the $\frac{3}{2}\,\frac{3}{2}$ final state interaction between $\pi^+$ and proton. This can be made writing the $R$ matrix element in the following way (**):

$$(3) \qquad R = R_0 + (S - 1) P_{33} R_0 ,$$

---

(*) See: I. M. JAUCH and F. ROHRLICH: *Theory of Photons and Electrons* (New York, 1955), pp. 163-167.

(**) See: B. F. TOUSCHEK: *Suppl. Nuovo Cimento*, **14**, 278 (1959).

where $R_0$ indicates the matrix element obtained in I and $P_{33}$ is the operator which projects the final state $|q_2 q_3 p_2\rangle = |\pi^-\pi^+ p\rangle$ on the state in which $\pi^+$ and p have relative angular momentum $J = \frac{3}{2}$.

The meaning of (3) is the following: for the production in the states $J \neq \frac{3}{2}$, where the operator $I - P_{33}$ is used, the outgoing particles are represented by plane waves since we assume no force acting out of the production region; in the remaining state, we take account of the « secondary » interaction by means of a factor $S$ which represents the distortion of the pion wavefunction in the neighbourhood of the proton.

This distortion may be calculated assuming the pion-nucleon forces arising from a short range potential, the characteristics of which may be referred to the $\alpha_{33}$ phase shift of scattering.

One gets (*)

$$S = \frac{3}{(qr_0)^3} \exp[i\alpha_{33}] \sin \alpha_{33}\,, \tag{4}$$

where $q$ is the c.m. momentum.

Assuming for $\alpha_{33}$ the Chew-Low formula

$$\frac{q^3}{\omega} \operatorname{cotg} \alpha_{33} = \frac{3}{4f^2}\left(1 - \frac{\omega}{\omega_R}\right),$$

where $f^2 = 0.083$, $\omega$ is the total c.m. energy minus the nucleon mass and $\omega_R$ in $\mu = 1$ units is 2.15, one has for $r_0^3$ the value 0.62 imposing that $\lim_{q \to 0} S = 1$.

For calculating (2), we shall calculate

$$|\langle f|M|i\rangle|^2 = |\langle f|M_0|i\rangle|^2 + |S-1|^2 |\langle f|P_{33}M_0|i\rangle|^2 + \tag{5}$$
$$+ 2\,\mathrm{Re}\,(S-1)\langle f|P_{33}M_0|i\rangle\langle f|M_0|i\rangle^*.$$

In the subsequent sections we shall calculate separately the various terms of (5).

### 3. – Projection of the $\frac{3}{2}\,\frac{3}{2}$ part of the matrix element.

We shall get first the expression of $\langle f|P_{33}M_0|i\rangle$. Since we are interested to the energy spectrum of $\pi^-$, it will be suitable to work in the system $\boldsymbol{K} + \boldsymbol{P}_1 = \boldsymbol{q}_2$, *i.e.* in the c.m. system of $\pi^+$ and proton.

---

(*) See: B. F. Touschek (loc. cit.).

If we call $q$ the common momentum in this system ($\boldsymbol{P}_2 = -\boldsymbol{q}_3 = \boldsymbol{q}$), it may be seen that, for fixed values of the energy and angle of the $\pi^-$ in the laboratory system ($u^2$, $r^2$, $d^2$ fixed), all the c.m. energies are determined as functions of $u^2$ and $r^2$. The orientation of $\boldsymbol{q}$ with respect to the fixed $\boldsymbol{K}$, $\boldsymbol{P}_1$, $\boldsymbol{P}_2$ is completely free.

Let us define the angles as in Fig. 3.

The system is intrinsic to the configuration of the fixed vectors and (1), (2), (3) form a cartesian left-handed system.

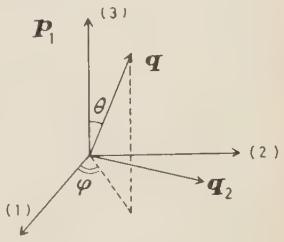

Fig. 3.

It follows that in this system the only free parameters in the matrix element $(f|M_0|i)$ are the direction of $\boldsymbol{q}$ and the spin $\sigma$ of the final proton, since now we are not interested in the $\gamma$ polarization and the spin of the initial proton.

Therefore we shall write the matrix element as $(\boldsymbol{q}, \sigma|M_0|i)$, where $|\boldsymbol{q}| = q$ is fixed as a function of $u^2$. It is convenient to replace the complete system $\boldsymbol{q}$, $\sigma$ with the new system $j$, $l$, $M$, $K$ where $j$ is the eigenvalue of the total angular momentum, $M$ its third component, $l$ the orbital momentum and $K$ the modulus of the momentum.

With these positions one has

$$P_{33} = \frac{1}{(2\pi)^3} \sum_{M=-\frac{3}{2}}^{\frac{3}{2}} \pi \int \mathrm{d}K \, |\tfrac{3}{2}\,1M\rangle\langle\tfrac{3}{2}\,1M| \,, \tag{6}$$

having here used the normalization

$$\langle jlMK|j'l'M'K'\rangle = (2\pi)^3 \, \delta(K-K') \, \delta_{jj'} \, \delta_{ll'} \, \delta_{MM'}$$

corresponding to one particle per unit volume.

In the calculations which follow we shall use the plane and spherical solutions of the Dirac equation given by AKHIEZER and BERESTETSKY (*). One gets

$$\langle \boldsymbol{q}, \sigma|P_{33}M_0|i\rangle = \sum_M v_\sigma^* \Omega_{\frac{3}{2}1M}(\boldsymbol{n}) \sum_\mu \int \mathrm{d}O' \, v_\mu \, \Omega^*_{\frac{3}{2}1M}(\boldsymbol{n}') \langle \boldsymbol{q}', \mu|M_0|i\rangle \,. \tag{7}$$

In (7) $O'$ is the solid angle of $\boldsymbol{n}' = \boldsymbol{q}'/q$ and it is meant $|\boldsymbol{q}'| = |\boldsymbol{q}| = q$; furthermore $v_\mu$ and $\Omega_{jlM}$ are defined as in (*).

---

(*) A. I. AKHIEZER and V. B. BERESTETSKY: *Quantum Electrodynamics*, English Translation by Technical Information Service, U.S.A., pp. 74-76, 85-95.

Using (7) we shall be able to evaluate all the terms of (5); the three contributions will be calculated separately in the subsequent sections.

## 4. – Contribution to the cross-section of the term $\langle q, \sigma | P_{33} M_0 | i \rangle$.

From (7) one has:

$$(8) \qquad |\langle \boldsymbol{q}, \sigma | P_{33} M_0 | i \rangle|^2 = \sum_{M,M'} v_\sigma^* \Omega_{\frac{3}{2}1M}(\boldsymbol{n}) \Omega^*_{\frac{3}{2}1M'}(\boldsymbol{n}') r_\sigma \cdot \sum_{\mu\mu'} \int \mathrm{d}O' \, \mathrm{d}O'' \, r_\mu \Omega^*_{\frac{3}{2}1M}(\boldsymbol{n}') \cdot \cdot \Omega_{\frac{3}{2}1M'}(\boldsymbol{n}'') v^*_{\mu'} \langle \boldsymbol{q}', \mu | M_0 | i \rangle \langle \boldsymbol{q}'', \mu' | M_0 | i \rangle^* .$$

If we want to have the differential cross-section for the $\pi^-$, we must insert (8) in (2) and integrate over the phase space of $\boldsymbol{P}_2$ and $\boldsymbol{q}_3$. In the c.m. system of these two vectors the integration is reduced to the one over the unit vector $|\boldsymbol{q}|/q$. Before performing the integration, we can rewrite (8) by means of the orthogonality of the spherical spinors and summing over final spins:

$$(9) \qquad \sum_\sigma |\langle \boldsymbol{q}, \sigma | P_{33} M_0 | i \rangle|^2 = \sum_M |\Omega_{\frac{3}{2}1M}(\boldsymbol{n})|^2 \cdot \sum_{\mu\mu'} \int \mathrm{d}O' \, \mathrm{d}O'' \, C^{1,M-\mu,\mu}_{\frac{3}{2}M} \cdot \cdot Y^*_{1,M-\mu}(O') C^{1,M-\mu',\mu'}_{\frac{3}{2}M} Y_{1,M-\mu'}(O'') \langle \boldsymbol{q}', \mu | M_0 | i \rangle \langle \boldsymbol{q}'', \mu' | M_0 | i \rangle^* .$$

We have now to insert the explicit expression of $\langle f | M_0 | i \rangle$, average over initial spins and sum over the two polarization states of the photon:

$$(9') \qquad \tfrac{1}{2} \sum_{\sigma'} \sum_\sigma \sum_i |\langle P_{33} M_0 \rangle|^2 = \tfrac{1}{2} \sum_M |\Omega_{\frac{3}{2}1M}(\boldsymbol{n})|^2 \sum_{\mu\mu'} \int \mathrm{d}O' \, \mathrm{d}O'' \, C^{1,M-\mu,\mu}_{\frac{3}{2}M} \cdot$$

$$\cdot Y^*_{1,M-\mu}(O') C^{1,M-\mu',\mu'}_{\frac{3}{2}M} Y_{1,M-\mu'}(O'') \sum_{\sigma'} \sum_i \langle \boldsymbol{q}', \mu | M_0 | \boldsymbol{P}, \sigma', e_{(i)} \rangle \langle \boldsymbol{q}'', \mu' | M_0 | \boldsymbol{P}, \sigma', e_{(i)} \rangle^* ,$$

where

$$(10) \qquad \sum_{\sigma',i} \langle M_0 \rangle \langle M_0 \rangle^* = F(w'^2) \frac{1}{q_1'^2 + \mu^2} \exp [i \delta(w'^2)] \cdot$$

$$\cdot \sum_i \varepsilon_{\lambda\mu\nu\varrho} q'_{1\mu} q'_{3\nu} q_{2\varrho} e_{(i)\lambda} (\varepsilon_{\lambda'\mu'\nu'\varrho'} q''_{1\mu'} q''_{3\nu'} q_{2\varrho'} e_{(i)\lambda'})^* \cdot$$

$$\cdot F(w''^2) \frac{1}{q_1''^2 + \mu^2} \exp [- i \delta(w''^2)] \sum_{\sigma'} \overline{u}_\mu(\boldsymbol{P}'_2) \gamma_5 u_{\sigma'}(\boldsymbol{P}_1) \left(\overline{u}_{\mu'}(\boldsymbol{P}''_2) \gamma_5 u_{\sigma'}(\boldsymbol{P}''_1)\right)^* .$$

In (10) for sake of simplicity we have omitted constant factors and energy denominators; the complete expression of $\langle M_0 \rangle$ will be used in the final formula.

Furthermore the term $\exp[i\delta(w^2)]$ has been extracted from $F(w^2)$, being this more convenient in calculations; in I only $F^2(w^2)$ appears and so there is no inconsistency in the symbols used.

Writing down explicitly the various parts of (9') as functions of the solid angles $O'$, $O''$ and integrating, we get

$$(11)\qquad \tfrac{1}{2}\sum_{\sigma,\sigma',i}|\langle P_{33}M_0\rangle|^2=\frac{1}{8p^2q^2}N_q^2N_p^2\sum_M|\Omega_{\frac{3}{2}1M}(\boldsymbol{n})|^2\sum_{\mu\mu'}C_{\frac{3}{2}M}^{1,M-\mu',\mu'}C_{\frac{3}{2}M}^{1,M-\mu,\mu}K_{\mu\mu'}^M=$$

$$=\frac{1}{8p^2q^2}N_q^2N_p^2|\Omega_{\frac{3}{2}1M}(\boldsymbol{n})|^2\cdot\{2K_{\frac{3}{2}\frac{3}{2}}^{\frac{3}{2}}+\tfrac{4}{3}K_{\frac{1}{2}\frac{1}{2}}^{\frac{1}{2}}+\tfrac{2}{3}K_{\frac{1}{2}\frac{1}{2}}^{-\frac{1}{2}}\},$$

where is

$$K_{\mu\mu'}^M=\int dO'\,dO''\;W(O',O'')\,Y_{1,M-\mu}^*(O')\,Y_{1,M-\mu'}(O'')\cdot V_{\mu\mu'}(O',O''),$$

$$W(O',O'')=f(\theta')f(\theta'')\,Z(O',O'')\,F(w'^2)\,F(w''^2),$$

$$N_q=\sqrt{\frac{1}{2}\left(1+\frac{m}{P_{20}}\right)},\qquad N_p=\sqrt{\frac{1}{2}\left(1+\frac{m}{P_{10}}\right)},$$

$$Z(O',O'')=\sum_i\varepsilon_{\lambda\mu\nu\varrho}\dots(\varepsilon_{\lambda'\mu'\nu'\varrho'}\dots)^*,$$

$$f(\theta)=\frac{1}{\cos\theta-a},\qquad a=\frac{2P_{10}P_{20}-2m^2+\mu^2}{2pq},$$

and $V_{\mu\mu'}(O',O'')$, which for simplicity is not explicitly written here, multiplied for $N_q^2N_p^2$ gives the sum over $\sigma'$ in (10); the symmetry properties of $V_{\mu\mu'}$ have allowed some simplifications in the calculation of (11).

In the evaluation of the integrals $F(w^2)$ has been approximated with a suitable parabola (*); in fact we have seen that this can be made without appreciable errors in the allowed range for $w^2$:

$$4\mu^2\leqslant w^2\leqslant(W-m)^2.$$

Moreover for $\exp[i\,\delta(w^2)]$ we have assumed a constant value equal to the one at the resonance.

---

(*) See Appendix.

## 5. – Contribution of the interference term.

The quantity of interest is from (5):

$$2\,\mathrm{Re}\,(S-1)\langle P_{33}M_0\rangle\langle M_0\rangle^* . \tag{12}$$

One can verify that, integrating over the solid angle of the phase space expression, this contribution is simply related to what we have already calculated. From (7):

$$\langle \boldsymbol{q},\sigma|P_{33}M_0|\boldsymbol{P},\sigma',i\rangle\langle \boldsymbol{q},\sigma|M_0|\boldsymbol{P},\sigma',i\rangle^* =$$

$$= \sum_M v^*_\sigma \Omega^*_{\frac{3}{2}1M}(\boldsymbol{n}) \sum_\mu \int \mathrm{d}O' v_\mu \Omega^*_{\frac{3}{2}1M}(\boldsymbol{n}')\langle \boldsymbol{q}',\mu|M_0|i\rangle\langle \boldsymbol{q},\sigma|M_0|i\rangle^* .$$

We have to sum over the final spins $\sigma$ and polarizations $i$, to average over the initial spins $\sigma'$ and to perform the above integration over the solid angle of $\boldsymbol{q}$. One has

$$\frac{1}{2}\sum_{\sigma\sigma' i}\int \mathrm{d}O\,\langle P_{33}M_0\rangle\langle M_0\rangle^* = \frac{1}{2}\sum_M\sum_{\sigma\mu} C^{1,M-\sigma,\sigma}_{\frac{3}{2}M} C^{1,M-\mu,\mu}_{\frac{3}{2}M}\int \mathrm{d}O'\,\mathrm{d}O''\cdot Y^*_{1,M-\mu}(\boldsymbol{n}')\cdot \tag{13}$$

$$\cdot\, Y_{1,M-\sigma}(\boldsymbol{n})\cdot\sum_{\sigma' i}\langle \boldsymbol{q}',\mu|M_0|\rangle\langle \boldsymbol{q},\sigma|M_0|\rangle^* ,$$

and using (11)

$$\frac{1}{2}\sum_{\sigma\sigma' i}\int \mathrm{d}O\,\langle P_{33}M_0\rangle\langle M_0\rangle^* = \frac{1}{8p^2q^2}\,N_q^2 N_p^2 \sum_M\sum_{\sigma\mu} C^{1,M-\sigma,\sigma}_{\frac{3}{2}M} C^{1,M-\mu,\mu}_{\frac{3}{2}M} K^M_{\mu\sigma} .$$

For the reasons given in the preceeding section, finally we have

$$\frac{1}{2}\sum_{\sigma\sigma' i}\int \mathrm{d}O\,\langle P_{33}M_0\rangle\langle M_0\rangle^* = \frac{1}{8p^2q^2}\,N_q^2 N_p^2\left\{2K^{\frac{3}{2}}_{\frac{1}{2}\frac{1}{2}} + \tfrac{4}{3}\,K^{\frac{1}{2}}_{\frac{1}{2}\frac{1}{2}} + \tfrac{2}{3}\,K^{-\frac{1}{2}}_{\frac{1}{2}\frac{1}{2}}\right\}, \tag{14}$$

and, being

$$\frac{1}{2}\sum_{\sigma\sigma' i}\int \mathrm{d}O\cdot 2\,\mathrm{Re}\,(s-1)\,\langle P_{33}M_0\rangle\langle M_0\rangle^* = 2\,\mathrm{Re}\,(s-1)\,\frac{1}{2}\sum_\sigma\int \mathrm{d}O'\sum_{\sigma' i}\langle P_{33}M_0\rangle\langle M_0\rangle^* ,$$

the calculation of the interference term, after having integrated over the solid angle, is

$$2\,\frac{1}{2}\sum_{\sigma\sigma' i}\int \mathrm{d}O\;\mathrm{Re}\,(s-1)\langle P_{33}M_0\rangle\langle M_0\rangle^* = \tag{15}$$

$$= 2\,\mathrm{Re}\,(s-1)\,\frac{1}{8p^2q^2}\,N_q^2 N_p^2\left\{2K^{\frac{3}{2}}_{\frac{1}{2}\frac{1}{2}} + \tfrac{4}{3}\,K^{\frac{1}{2}}_{\frac{1}{2}\frac{1}{2}} + \tfrac{2}{3}\,K^{-\frac{1}{2}}_{\frac{1}{2}\frac{1}{2}}\right\}.$$

## 6. – Contribution to the uncorrected term.

Remembering the matrix element given in I,

$$\frac{1}{2}\sum_{\sigma\sigma' i}|\langle M_0\rangle|^2 = \frac{1}{2}\frac{F^2(w^2)}{[(P_2-P_1)^2+\mu^2]^2}\sum_i \varepsilon_{\lambda\mu\nu\varrho}K_\mu q_{3\nu}q_{2\varrho}e_{\lambda(i)}\cdot \tag{16}$$

$$\cdot(\varepsilon_{\lambda'\mu'\nu'\varrho'}K_{\mu'}q_{3\nu'}q_{2\varrho'}e_{\lambda'(i)})^*\sum_{\sigma\sigma'}|\bar{u}_\sigma(\boldsymbol{P}_2)\gamma_5 u_{\sigma'}(\boldsymbol{P}_1)|^2,$$

where the constants and energy denominators have been omitted. Performing as in I the summations over spins and polarizations and integrating over the solid angle,

$$\int \mathrm{d}O\,\frac{1}{2}\sum_{\sigma\sigma' i}|\langle M_0\rangle|^2 = \frac{2\pi}{8p^2q^2P_{10}P_{20}}Q\{2qq_{30}(B_2J_3+B_3J_7)+ \tag{17}$$

$$+q^2[2C_2J_4+C_3J_8+\tfrac{1}{2}D_2(J_{10}-J_1)]+B_5J_6\}-\frac{2\pi}{8p^2q^2P_{10}P_{20}}pq\cdot$$

$$\cdot\{2qq_{30}(B_2J_4+B_4J_8)+q^2[2C_2J_5+C_3J_9+\tfrac{1}{2}D_2(J_{11}-J_2)]+B_5J_7\},$$

where $Q=P_{10}P_{20}-m^2$ and $F^2(w^2)$ has been approximated with a suitably chosen parabola. The remaining quantities introduced in (17) are given explicitely in the Appendix.

## 7. – Energy spectrum and angular cross-section.

Now (11), (15), (17) allow us to get an explicit expression of (5) and then to evaluate the cross-section. All the above calculations have been made in the c.m. system $\boldsymbol{p}_2+\boldsymbol{q}_3=0$.

We have already pointed out that all the energies and angles between $\boldsymbol{K}$, $\boldsymbol{p}_1$, $\boldsymbol{q}_2$ are depending only from $u^2$ and $r^2$; for fixed values of these two invariants (*i.e.* $\pi^-$ energy and angle determined in the laboratory system), the remaining variability in $\Delta^2$, $w^2$ gives the freedom of orientation of $\boldsymbol{q}=\boldsymbol{p}_2=-\boldsymbol{q}_3$.

Having integrated over its solid angle, we have in practice integrated over $\Delta^2$ and $w^2$, *i.e.* over the degrees of freedom which do not regard the particle ($\pi^-$) of which we want the cross-section.

Let us write

$$\begin{cases}\displaystyle\int \mathrm{d}O\,\sum|\langle M_0\rangle|^2 = F_1(u^2,r^2)\,,\\ \displaystyle\int \mathrm{d}O\,\sum|\langle P_{33}M_0\rangle|^2 = F_2(u^2,r^2)\,.\end{cases} \tag{18}$$

Then, after some standard manipulations and reintroducing the factors dropped above:

$$(19)\qquad \frac{\partial^2\sigma}{\partial u^2\,\partial r^2} = \frac{0.007\,8}{(2\pi)^4 K^2 W^2}\,\frac{g^2\gamma^2}{\mu^6}\cdot$$
$$\cdot\left\{\frac{qP_{10}P_{20}}{u}\,[F_1(u^2, r^2) + (|s-1|^2 + 2\,\mathrm{Re}\,(s-1))\,F_2(u^2, r^2)\right\},$$

where $K$ is the $\gamma$ momentum in the total c.m. system ($\boldsymbol{K}+\boldsymbol{p}_1 = 0$) and $W$ is the c.m. energy already defined. Since the $\pi^-$ total energy, the kinetic energy and the angle with the forward direction have the expressions

$$(20)\qquad \begin{cases} q^L_{20} = \dfrac{r^2 + m^2 + \mu^2}{2m}, \qquad T^L_{\pi^-} = \dfrac{r^2}{2m} + \dfrac{(m-\mu)^2}{2m}, \\ \cos\alpha_L = \cos \boldsymbol{K}_L \boldsymbol{q}_{2L} = -\dfrac{W^2 - r^2 - u^2 - m^2}{K_{0L}\,|\boldsymbol{q}_{2L}|} + \dfrac{q^L_{20}}{|\boldsymbol{q}_{2L}|}, \end{cases}$$

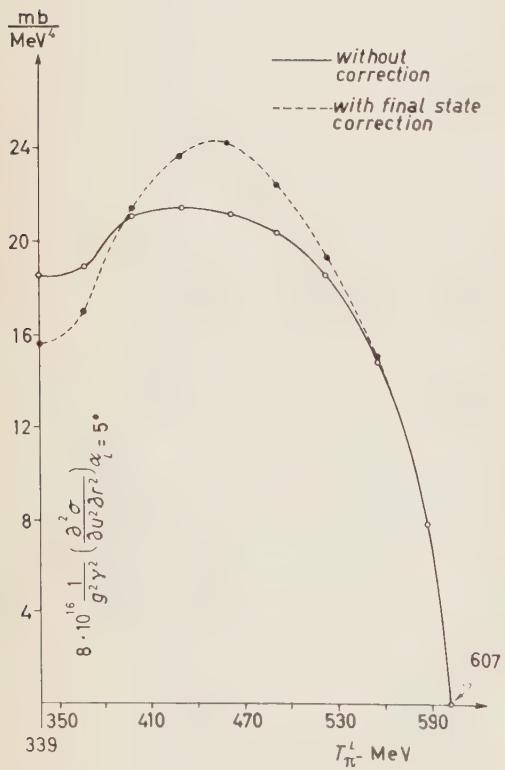


Fig. 4.

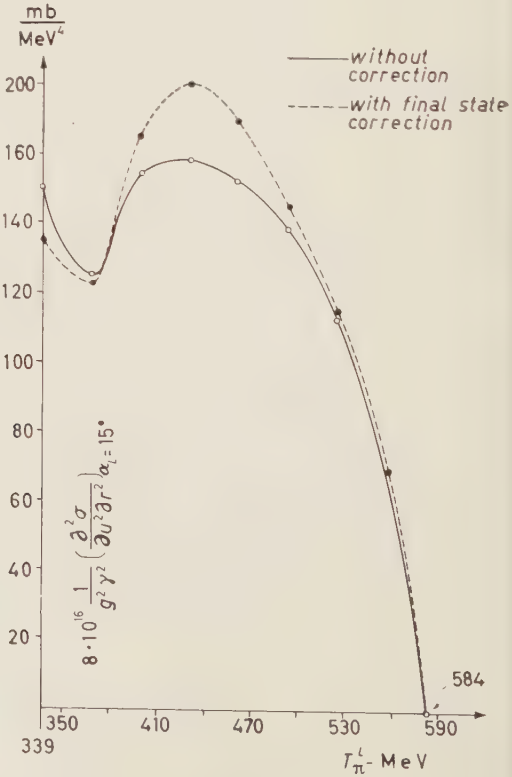


Fig. 5.

as functions of the invariants $u^2$, $r^2$, from (19) we may get the energy spectrum and angular distribution of the $\pi^-$ in the laboratory system. In fact it is

$$\frac{\partial^2\sigma}{\partial T^L_{\pi^-}\partial\cos\alpha_L} = 4mK_{0L}|\boldsymbol{q}_{2L}|\frac{\partial^2\sigma}{\partial u^2\,\partial r^2}\,.$$

Now we shall use (19) for calculating the $\pi^-$ energy spectrum at three laboratory fixed angles.

If we want that the assumptions we have made for our model be fulfilled, we must cut the small $\pi^-$ energies corresponding to high proton momentum transfers.

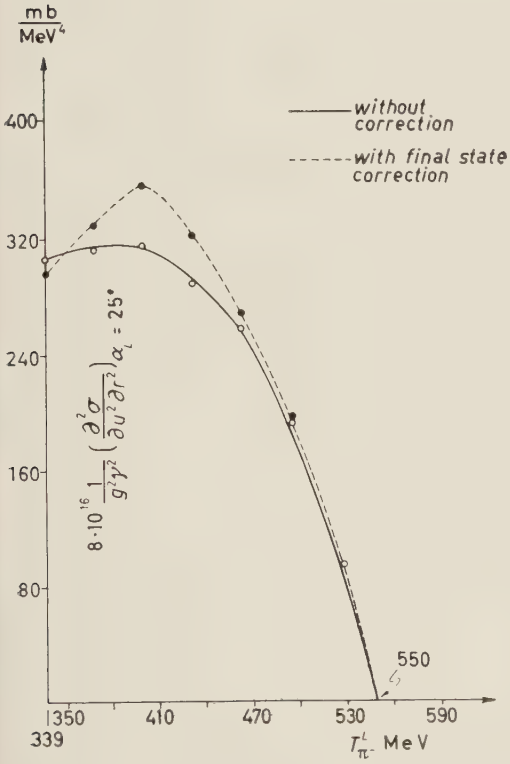


Fig. 6.

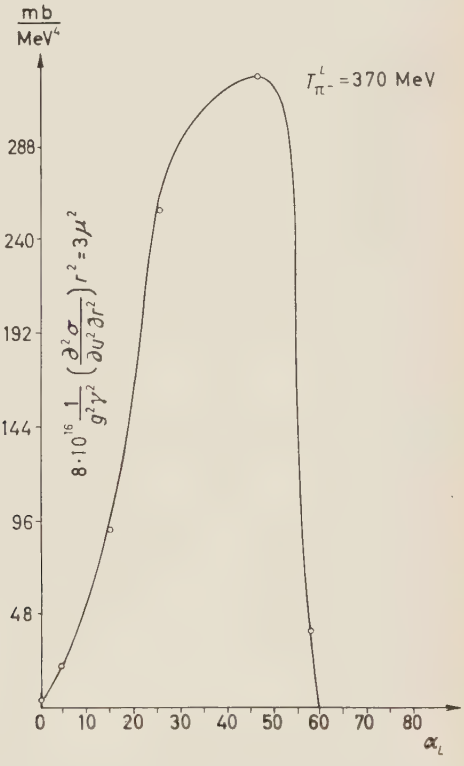


Fig. 7.

Assuming $T_{2L}\leqslant 140$ MeV, one has that it must be $T^L_{\pi^-}\geqslant 339$ MeV. If we want to have at our disposal an energy range sufficiently large, small angles must be considered: we shall calculate (19) for 900 MeV $\gamma$ at the following angles: $\alpha_L = 5°, 15°, 25°$. The obtained curves are drawn in Fig. 4, 5, 6 with broken

lines; in the same figures they are compared with $\partial^2\sigma/\partial u^2\,\partial r^2$ obtained without considering the final state interaction (curves with full lines).

The latter ones have been calculated inserting in (2) only the term given by (17), *i.e.* in the case of $S=1$.

It may be seen that the correction is at most of the order of $(14\div16)\,\%$. The behaviour of the cross-section is not essentially changed, being the maximum of the corrections in correspondence of the maximum of the curves. The corrections are therefore very small and hard to appreciate experimentally.

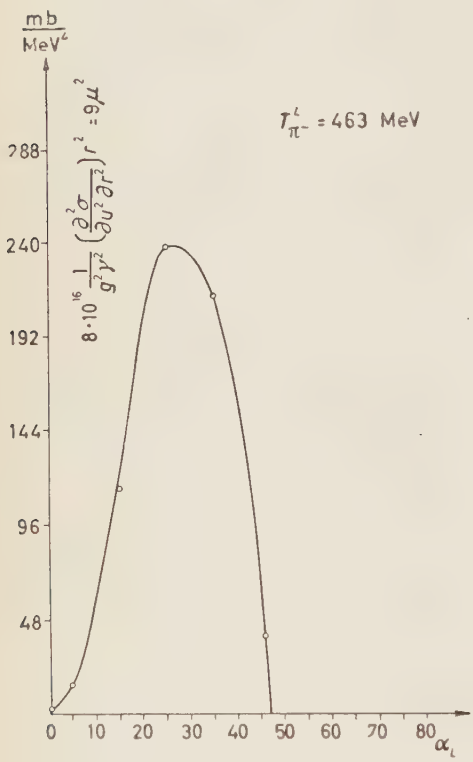


Fig. 8.

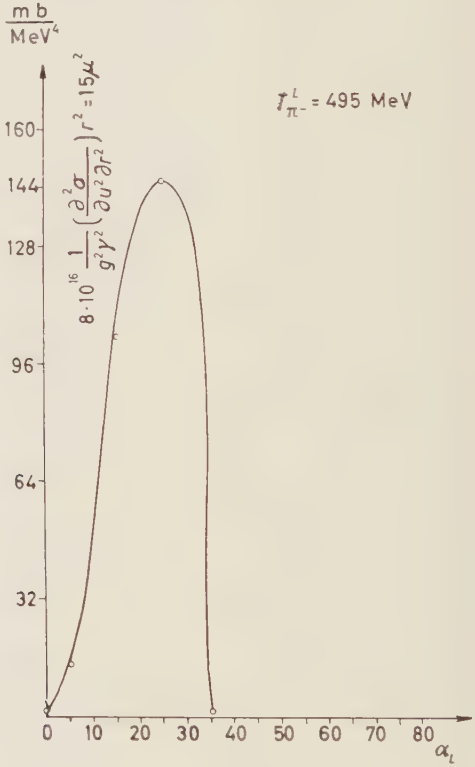


Fig. 9.

Moreover, as already noted in I for the proton energy spectrum the behaviour of the cross-section does not depend on the actual form of the pion-pion interaction; in fact if we assume $F^2(u^2)=1.35$ we can obtain the same curves with full lines of Fig. 4, 5, 6 within an error of 8 %. For this reason we shall calculate the $\pi^-$ angular cross-section taking into account only the uncorrected term and assuming a constant value for $F^2(w^2)$. In Fig. 7, 8, 9 $\partial^2\sigma/\partial u^2\,\partial r^2$ is drawn at $T_{\pi^-}^L=370$, 463, 495 MeV.

## 8. – Conclusions.

As already said in the introduction, the aim of this calculation was to see if our diagram gives results experimentally distinguishable from the Drell's ones. Let us refer to the angular distributions of Fig. 7, 8, 9 which have a more characteristic behaviour than the energy spectra. The corresponding angular cross-section evaluation by means of the Drell model shows a sharp maximum about 11° which is not displaced changing $T_{\pi^-}^L$: On the contrary our curves have a maximum which is displaced towards small angles at increasing $T_{\pi^-}$: specially at the smallest energy (Fig. 7) it is well separated from the Drell's one.

But to compare also the importance of the two diagrams it is necessary to determine the constant $\gamma$ in our formula. We have tried to obtain an approximate value for it making use of the data obtained by De Tollis, Ferrari and Munczek (*) in their check of the importance of the dipion model in the photoproduction (as an average value of their results we have assumed their constant $A/e \cdot m \cdot f \sim 1.5$).

Making use also of the results given by Gourdin, Lurié and Martin (**) and Bowcock, Cottingham and Lurié (***), we obtain $\gamma \sim 0.85$.

Being $g^2 = 4\pi f^2 (4m^2/\mu^2) \sim 180$, we have $\gamma^2 g^2 \sim 149$; with this value for $\gamma^2 g^2$, the peaks of the curves of Fig. 7, 8, 9 result of the same order of magnitude of the corresponding Drell's peaks.

An experiment which measures the $\pi^-$ angular cross-sections at the energies we have chosen perhaps to supply a more precise determination of our constant.

A recent experiment by Walker *et al.* (****) cannot be checked with our calculations having used 1230 MeV $\gamma$. We have assumed a $\gamma$ energy of 900 MeV in view of an experiment to be performed using the Frascati synchrotron.

* * *

It is a pleasure to thank Prof. M. Cini for having suggested this calculation and for many useful discussions. As also thank Dr. E. Ferrari and Prof. A. Stanghellini for many stimulating remarks.

---

(*) B. De Tollis, E. Ferrari and H. Munczek: *Nuovo Cimento*, **18**, 198 (1960).
(**) M. Gourdin, D. Lurié and A. Martin: *Nuovo Cimento*, **18**, 933 (1960).
(***) J. Bowcock, W. N. Cottingham and D. Lurié: *Phys. Rev. Lett.*, **5**, 386 (1960).
(****) J. R. Kilner, R. E. Diebold and R. L. Walker: *Phys. Rev. Lett.*, **5**, 518 (1960).

## Appendix

In this appendix we report some details of our calculations.
Writing $w^2$ as

$$w^2 = \varrho + \lambda \sin\theta \sin\varphi + \chi \cos\theta \,, \tag{A.1}$$

where

$$\varrho = 2q_{20}q_{30} + 2\mu^2; \quad \lambda = 2\,|\boldsymbol{q}_3|\,|\boldsymbol{q}_2|\sin\widehat{\boldsymbol{q}_2\boldsymbol{p}_1}; \quad \chi = 2\,|\boldsymbol{q}_3|\,|\boldsymbol{q}_2|\cos\widehat{\boldsymbol{q}_2\boldsymbol{p}_1}\,,$$

and representing $F(w^2)$ with the parabola

$$F(w^2) \simeq \alpha w^4 + \beta w^2 + \gamma \,, \tag{A.2}$$

($\alpha = -0.00484$, $\beta = 0.155$, $\gamma = -0.038$); we have also

$$F(w^2) = \mathrm{d}\sin^2\theta \exp[2i\varphi] + \mathrm{d}\sin^2\theta \exp[-2i\varphi] - \\ - E\exp[i\varphi] + E\exp[-i\varphi] + G\,. \tag{A.3}$$

The symbols in (A.3) have the meaning

$$d = -\frac{\alpha\lambda}{4}\,,$$

$$\begin{cases} E = e_1 \sin\theta + e_2 \sin\theta\cos\theta & \begin{cases} e_1 = i/2(2\alpha\varrho\lambda + \beta\lambda)\,, \\ e_2 = i\alpha\chi\lambda\,, \end{cases} \\ G = g_1\cos^2\theta + g_2\cos\theta + g_3 & \begin{cases} g_1 = \alpha(\chi^2 - \frac{1}{2}\lambda^2)\,, \\ g_2 = 2\alpha\varrho\chi + \beta\chi\,, \\ g_3 = \alpha\varrho^2 + \beta\varrho + \gamma + (\alpha/2)\lambda^2\,. \end{cases} \end{cases} \tag{A.4}$$

In the same manner, also $F^2(w^2)$ has been approximated with

$$F^2(w^2) \simeq \alpha' w^4 + \beta' w^2 + \gamma' \,, \tag{A.5}$$

where $\alpha' = -0.00823$, $\beta' = 0.263$, $\gamma' = -0.665$.

It can be repeated what has been done in (A.3) and (A.4) using primed symbols.

Then the quantities in (17) have the meaning:

$$J_n = \begin{cases} \mathrm{d}'I(\sin^4\theta\cos^{n-1}\theta)\,, & n = 1, 2 \\ -iI(\sin\theta\, E'\cos^{n-3}\theta)\,, & n = 3, 4, 5, \\ I(G'\cos^{n-6}\theta)\,, & n = 6, 7, 8, 9, \\ I(\sin^2\theta\, G'\cos^{n-10}\theta)\,, & n = 10, 11, \end{cases} \tag{A.6}$$

where

$$I[f(\theta)] = \int_0^\pi \frac{f(\theta)}{(\cos\theta - a)^2} \sin\theta \, \mathrm{d}\theta . \tag{A.7}$$

Furthermore:

$$\begin{cases} B_0 = q_2^2 K_0^2 - 2K_0 q_{20}(Kq_2) , \\ B_2 = K_{12}[q_2^2 K_0 - K_0(Kq_2) - q_{20}(Kq_2)] , \\ B_3 = q_2^2 K_{13}^2 - K_0(Kq_2)K_{23} - q_{20}(Kq_2)K_{13} , \\ B_4 = -(Kq_2)^2 + B_0 , \\ B_5 = q_3^2(Kq_2)^2 + q_{30}^2 B_0 , \\ C_2 = [q_2^2 K_{13} - K_{23}(Kq_2) - K_{13}(Kq_2)]K_{12} , \\ C_3 = -K_{13}[2K_{23}(Kq_2) + K_{13}] , \\ C_4 = C_3 + (Kq_2)^2 , \\ D_2 = q_2^2 K_{12}^2 - 2K_{22}(Kq_2)K_{12} , \end{cases} \tag{A.8}$$

where $K_{1i}$, $K_{2i}$ $(i = 1, 2, 3)$ are the $i$-components of $\boldsymbol{K}$ and $\boldsymbol{q}_2$ with respect to the system of Fig. 3.

***Note added in proof.***

In the paper by D. Boccaletti and C. Gualdi: *Nuovo Cimento*, **18**, 895 (1960), the strong peak at 15 MeV in the proton energy spectra is due to an error in numerical computations. The correct spectra show a smoother peak at an energy between 60 and 90 MeV.


## RIASSUNTO

Facendo uso dell'elemento di matrice per la fotoproduzione doppia di pioni carichi costruito nella prima parte di questo lavoro, vengono calcolati lo spettro energetico del $\pi^-$ a tre angoli fissati e la distribuzione angolare della stessa particella a tre energie fissate. Si è anche tenuto conto dell'interazione nello stato finale $\frac{3}{2}\,\frac{3}{2}$ fra $\pi^+$ e protone: la correzione risultante cambia molto poco la forma dello spettro del $\pi^-$. Lo scopo di questo calcolo è di vedere se è possibile separare sperimentalmente le previsioni del nostro diagramma da quelle del diagramma « alla Drell », essendo nulla l'interferenza dei due grafici. Si prende 900 MeV per l'energia del $\gamma$.

IL NUOVO CIMENTO Vol. XX, N. 2 16 Aprile 1961

# Scattering by a Given Class of Non-Central Forces.

R. Vinh Mau and A. Martin

*CERN - Geneva*

(ricevuto il 3 Marzo 1961)

**Summary.** — Previous work has shown the remarkably simple properties of the partial wave scattering amplitude and corresponding radial functions derived from a potential, which is a continuous superposition of exponential or Yukawa potentials. Here we extend these results to the case where non-central forces are present in the scattering of two spin one half particles. We give a method of calculation of the $S$-matrix elements in terms of the inverse Laplace transforms of the various radial potentials, we establish the analytic properties of these matrix elements and get integral representations of the wave functions.

## 1. – Introduction.

The analytic properties of the partial wave scattering amplitudes have been established recently by one of us (A.M.) [1] for the following class of *central* potentials

$$V(r) = \int_{\mu}^{+\infty} C(\alpha) \exp[-\alpha r] \, \mathrm{d}\alpha \,, \qquad |C(\alpha)| < M\alpha^{1-\varepsilon} \,. \tag{1}$$

In this paper, we wish to consider the properties of the radial wave functions of the Schrödinger equation for two spin $\frac{1}{2}$ particles in the presence of *non-central* potentials:

$$V(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2; r) = V_d(r) + V_\sigma(r) \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + V_T(r) S_{12} + V_0(r) \boldsymbol{L} \cdot \boldsymbol{S} \,. \tag{2}$$

[1] A. Martin: *Nuovo Cimento*, **14**, 403 (1959); **15**, 99 (1960).

where the position-dependent parts $V_d$, $V_\sigma$, $V_T$, $V_0$ belong to the family (1).

At first sight, the extension of the methods used in [1] might be complicated by the fact that the radial equations are now coupled, but we shall see that we may still solve the problem by using procedures which are very close to those described in [1], provided a special choice of independent solutions of the Schrödinger equation is made.

In Sections **2** and **3**, we shall derive the expressions of the $S$-matrix, the eigenphase-shifts and the admixture parameters in terms of special types of irregular solutions. In Section **4**, these solutions are shown to be directly obtained from the inverse Laplace transforms of the potentials by an iteration method. Finally, the analytic properties of the $S$-matrix are studied in Section **5**.

## 2. – Expression of the $S$-matrix in the JLSM scheme.

If we separate the Schrödinger equation (*e.g.* describing the $N$-$P$ system interacting through $V(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2; \boldsymbol{r})$) into partial waves according to the $JLSM$ scheme, it is well-known [2,3] that, for the $L = J-1$ and $L = J+1$ triplet states, the radial wave functions $u_J$ and $w_J$ are mixed by the following system of coupled differential equations:

$$
(3)\qquad \begin{cases} u_J''(r) + \left[k^2 - \dfrac{(J-1)J}{r^2} - V_\sigma(r) - (J-1)\,V_0(r) - \dfrac{2(J-1)}{2J+1}\,V_T(r)\right] u_J = \\ \qquad\qquad = -\dfrac{6\sqrt{J(J+1)}}{2J+1}\,V_T(r) w_J\,, \\ w_J''(r) + \left[k^2 - \dfrac{(J+1)(J+2)}{r^2} - V_\sigma(r) + (J+2)\,V_0(r) + \dfrac{2(J+2)}{2J+1}\,V_T(r)\right] w_J = \\ \qquad\qquad = -\dfrac{6\sqrt{J(J+1)}}{2J+1}\,V_T(r) u_J\,, \end{cases}
$$

where $V_\sigma = V_d + V_\sigma$.

A general solution of (3) may be written as

$$
(4)\qquad \begin{pmatrix} u_J(r) \\ w_J(r) \end{pmatrix} = a_1 \psi_1^+ + a_2 \psi_2^+ + b_1 \psi_1^- + b_2 \psi_2^-\,,
$$

where the $\psi_\lambda^{+,-}(r, k)$, $(\lambda = 1, 2)$, are a complete set of linearly independent so-

(2) W. Rarita and J. Schwinger: *Phys. Rev.*, **59**, 556 (1941).
(3) R. Newton: *Phys. Rev.*, **100**, 412 (1955).

lutions of (3) defined by:

$$(5) \qquad \psi_\lambda^+ = \begin{pmatrix} f_\lambda(r, \quad k) \exp[ikr] \\ g_\lambda(r, -k) \exp[ikr] \end{pmatrix}, \qquad \psi_\lambda^- = \begin{pmatrix} f_\lambda(r, k) \exp[-ikr] \\ g_\lambda(r, k) \exp[-ikr] \end{pmatrix},$$

with the following asymptotic behaviour for $r \to \infty$:

$$(6) \qquad \begin{cases} \psi_1^+ \sim \begin{pmatrix} \exp[ikr] \\ \exp[ikr] \end{pmatrix}, & \psi_2^+ \sim \begin{pmatrix} \exp[ikr] \\ -\exp[ikr] \end{pmatrix}, \\ \psi_1^- \sim \begin{pmatrix} \exp[-ikr] \\ \exp[-ikr] \end{pmatrix}, & \psi_2^- \sim \begin{pmatrix} \exp[-ikr] \\ -\exp[-ikr] \end{pmatrix}. \end{cases}$$

The question of existence and determination of this complete set will be examined in Section **4**.

Now, if we choose from (4) the solutions $(u_a, w_a)$ and $(u_b, w_b)$ such that their respective asymptotic form contains only pure $L = J - 1$ or $L = J + 1$ incoming waves, *i.e.*

$$\begin{pmatrix} u_a \\ w_a \end{pmatrix} = a_1 \psi_1^+ + a_2 \psi_2^+ + c(\psi_1^- + \psi_2^-),$$

$$\begin{pmatrix} u_b \\ w_b \end{pmatrix} = a_1' \psi_1^+ + a_2' \psi_2^+ + c'(\psi_1^- - \psi_2^-),$$

Then the $S$-matrix

$$S = \begin{pmatrix} S_{J-1,J-1} & S_{J-1,J+1} \\ S_{J+1,J-1} & S_{J+1,J+1} \end{pmatrix},$$

may be easily obtained:

$$(7) \qquad \begin{cases} S_{J-1,J-1} = -\dfrac{a_1 + a_2}{2c} & S_{J-1,J+1} = -\dfrac{a_1' + a_2'}{2c'} \\ S_{J+1,J-1} = \dfrac{a_2 - a_1}{2c} & S_{J+1,J+1} = \dfrac{a_2' - a_1'}{2c'}, \end{cases}$$

the $a_i$ and $a_i'$ are determined by the regularity condition (*):

$$\lim_{r \to 0} \begin{pmatrix} r^{J-1} u_{a,b} \\ r^{J+1} w_{a,b} \end{pmatrix} = 0,$$

(*) It is known that this necessary condition is also sufficient to make $u$ and $w$ vanish at the origin provided the potentials are less singular than $1/r^2$, which is the case here.

which gives rise to the systems:

$$(8)\qquad \begin{cases} a_1 f_1(-k) + a_2 f_2(-k) + c\big(f_1(k) + f_2(k)\big) = 0 \\ a_1 g_1(-k) + a_2 g_2(-k) + c\big(g_1(k) + g_2(k)\big) = 0 \end{cases}$$

$$(8')\qquad \begin{cases} a_1' f_1(-k) + a_2' f_2(-k) + c'\big(f_1(k) - f_2(k)\big) = 0 \\ a_1' g_1(-k) + a_2' g_2(-k) + c'\big(g_1(k) - g_2(k)\big) = 0 , \end{cases}$$

where (*)

$$f_\lambda(k) = \lim_{r\to 0} r^{J-1} f_\lambda(r, k) ,$$

$$g_\lambda(k) = \lim_{r\to 0} r^{J+1} g_\lambda(r, k) .$$

If we normalize the incoming waves so that $c = c' = 1$, we can readily show that the $S$-matrix is symmetric: $S_{J+1,J-1} = S_{J-1,J+1}$, by means of (8), (8') and the following relation:

$$(9)\qquad f_1(k) g_1(-k) - f_1(-k) g_1(k) + f_2(k) g_2(-k) - f_2(-k) g_2(k) = 0$$

which will be proved in the Appendix.

### 3. – Eigenstates and eigenphase-shifts.

Though all the information needed is contained in the non-diagonal $S$-matrix obtained in Section 2, it may be worth-while to look at some direct way of obtaining the eigenphase-shifts $\delta_J^\alpha$, $\delta_J^\gamma$ (**) and the admixture parameters $\eta_J^\alpha$, $\eta_J^\gamma$ of the « eigenstates »: $(u_J^\alpha, w_J^\alpha)$ and $(u_J^\gamma, w_J^\gamma)$ defined by the following boundary conditions:

$$(A)\qquad \begin{cases} u_J^{\alpha,\gamma} \sim 2i \sin (kr + \delta_J^{\alpha,\gamma}) , \\ w_J^{\alpha,\gamma} \sim 2i\, \eta^{\alpha,\gamma} \sin (kr - \pi + \delta_J^{\alpha,\gamma}) . \end{cases} \qquad (r \to \infty).$$

$$(B)\qquad \lim_{r\to 0} \begin{cases} r^{J-1} u^{\alpha,\gamma} \to 0 , \\ r^{J+1} w^{\alpha,\gamma} \to 0 . \end{cases}$$

$$\text{Remembering } \begin{pmatrix} u \\ w \end{pmatrix} = a_1 \psi_1^+ + a_2 \psi_2^+ + b_1 \psi_1^- + b_2 \psi_2^- ,$$

(*) This procedure is only valid if the tensor force is not singular. See note added in proof.

(**) Note that in order to simplify the notations the eigenphase shifts are defined here as the sum of the usual eigenphase-shifts and $(J - 1)\pi/2$.

the asymptotic behaviour (A) tells us that:

$$(10)\qquad \begin{cases} b_1 = -a_1^* = -r_1 \exp[-i\delta_J^{\alpha,\gamma}] \\ b_2 = -a_2^* = -r_2 \exp[-i\delta_J^{\alpha,\gamma}] , \\ \eta^{\alpha,\gamma} = \dfrac{a_2 - a_1}{a_2 + a_1} . \end{cases}$$

The regularity condition (B) leads to the following system for real energies:

$$(10')\qquad \begin{cases} r_1[\cos\delta\ \mathrm{Im}\, f_1(k) + \sin\delta\ \mathrm{Re}\, f_1(k)] + r_2[\cos\delta\ \mathrm{Im}\, f_2(k) + \sin\delta\ \mathrm{Re}\, f_2(k)] = 0 , \\ r_1[\cos\delta\ \mathrm{Im}\, g_1(k) + \sin\delta\ \mathrm{Re}\, g_1(k)] + r_2[\cos\delta\ \mathrm{Im}\, g_2(k) + \sin\delta\ \mathrm{Re}\, g_2(k)] = 0 , \end{cases}$$

where $\delta$ is the common phase of $a_1$ and $a_2$. The compatibility condition for the system (10′) gives rise to the quadratic equation:

$$\frac{\mathrm{Im}\, f_1(k) + \mathrm{tg}\,\delta\ \mathrm{Re}\, f_1(k)}{\mathrm{Im}\, g_1(k) + \mathrm{tg}\,\delta\ \mathrm{Re}\, g_1(k)} = \frac{\mathrm{Im}\, f_2(k) + \mathrm{tg}\,\delta\ \mathrm{Re}\, f_2(k)}{\mathrm{Im}\, g_2(k) + \mathrm{tg}\,\delta\ \mathrm{Re}\, g_2(k)} ,$$

the roots of which determine the eigenphase-shifts $\delta^\alpha$ and $\delta^\gamma$; then the admixture parameters $\eta^\alpha$ and $\eta^\gamma$ may be obtained by means of (10′) and (10). Furthermore, from the relation (9) between the $f_\lambda$ and $g_\lambda$, one can deduce the well-known relation: $\eta^\alpha\eta^\gamma = -1$.

At this stage the problem is therefore to solve the coupled differential equations satisfied by $f_\lambda(r, k)$ and $g_\lambda(r, k)$, in which the exponential factors disappear, because of special choice of solutions:

$$(11)\qquad \begin{cases} f_\lambda'' - 2ikf_\lambda' - \left[\dfrac{(J-1)J}{r^2} + U_M^f\right] f_\lambda = U_T g_\lambda \\ g_\lambda'' - 2ikg_\lambda' - \left[\dfrac{(J+1)(J+2)}{r^2} + U_M^g\right] g_\lambda = U_T f_\lambda \end{cases} .$$

where

$$U_M^f(r) \equiv V_0(r) + (J-1)\left[V_0(r) + \frac{2}{2J+1} V_T(r)\right],$$

$$U_M^g(r) \equiv V_0(r) - (J+2)\left[V_0(r) + \frac{2}{2J+1} V_T(r)\right],$$

$$U_T(r) \equiv -\frac{6\sqrt{J(J+1)}}{2J+1} V_T(r) .$$

**4. – Explicit determination of $f_\lambda(r, k)$ and $g_\lambda(r, k)$.**

For sake of simplicity, let us first treat the special case $J=1$ ($^3S_1$ and $^3D_1$ states), then (11) becomes

$$\text{(12)} \qquad \begin{cases} f''_\lambda - 2ikf'_\lambda - V_C f_\lambda = U_T g_\lambda \\ g''_\lambda - 2ikg'_\lambda - \left(\dfrac{6}{r^2} + U_M\right) g_\lambda = U_T f_\lambda . \end{cases}$$

Now, because of the assumption

$$V_C(r) = \int_\mu^{+\infty} C_C(\alpha) \exp[-\alpha r]\, \mathrm{d}\alpha ,$$

$$U_{T,M}(r) = \int_\mu^{+\infty} C_{T,M}(\alpha) \exp[-ar]\, \mathrm{d}\alpha ,$$

and because of the asymptotic behaviour (6) of $f_\lambda(r, k)$ and $g_\lambda(r, k)$, it is natural to write

$$f_\lambda(r, k) = 1 + \int_0^{+\infty} \varrho_{\lambda,k}(\alpha) \exp[-\alpha r]\, \mathrm{d}\alpha ,$$

$$g_\lambda(r, k) = \delta_{\lambda 1} - \delta_{\lambda 2} + \int_0^{+\infty} \sigma_{\lambda,k}(\alpha) \exp[-\alpha r]\, \mathrm{d}\alpha .$$

Then it follows from (12)

$$\text{(13)} \qquad \begin{cases} \alpha(\alpha+2ik)\varrho_{1,k}(\alpha) = C_C(\alpha) + C_T(\alpha) + \displaystyle\int_0^{\alpha-\mu} \mathrm{d}\beta [C_C(\alpha-\beta)\varrho_{1,k}(\beta) + C_T(\alpha-\beta)\sigma_{1,k}(\beta)] , \\ \alpha(\alpha+2ik)\sigma_{1,k}(\alpha) - 6\alpha - 6\displaystyle\int_0^{\alpha} (\alpha-\beta)\sigma_{1,k}(\beta)\, \mathrm{d}\beta = C_M(\alpha) + C_T(\alpha) + \\ \qquad + \displaystyle\int_0^{\alpha-\mu} \mathrm{d}\beta [C_M(\alpha-\beta)\sigma_{1,k}(\beta) + C_T(\alpha-\beta)\varrho_{1,k}(\beta)] , \end{cases}$$

$$(13')\quad \begin{cases} \alpha(\alpha+2ik)\varrho_{2,k}(\alpha)=C_c(\alpha)+C_T(\alpha)+\int_0^{\alpha-\mu} d\beta[C_c(\alpha-\beta)\varrho_{2,k}(\beta)+C_T(\alpha-\beta)\sigma_{2,k}(\beta)]\,, \\ \alpha(\alpha+2ik)\sigma_{2,k}(\alpha)+6\alpha-6\int_0^{\alpha}(\alpha-\beta)\sigma_{2,k}(\beta)\,d\beta=-C_M(\alpha)-C_T(\alpha)+ \\ \qquad +\int_0^{\alpha-\mu} d\beta[C_M(\alpha-\beta)\sigma_{2,k}(\beta)+C_T(\alpha-\beta)\varrho_{2,k}(\beta)]. \end{cases}$$

We are going now to show that these systems (13) and (13′) may be recast in such a form that they can be solved by an iteration procedure.

First, by introducing the auxiliary function

$$y(\alpha)=1+\int_0^{\alpha}\sigma_{1,k}(\beta)\,d\beta\,,$$

the 2-nd integral equation of (13) may be transformed into the following inhomogeneous differential equation:

$$(14)\qquad \alpha(\alpha+2ik)y''+2(\alpha+ik)y'-6y=F'(\alpha)\,,$$

with

$$F(\alpha)=C_M(\alpha)+C_T(\alpha)+\int_0^{\alpha-\mu} d\beta[C_M(\alpha-\beta)\sigma_{1,k}(\beta)+C_T(\alpha-\beta)\varrho_{1,k}(\beta)]\,.$$

The associated homogeneous equation is a Legendre equation, so the solution of (14) satisfying the right boundary condition for $\alpha<\mu$ ($F(\alpha)=0$ in this range) is

$$(15)\qquad y(\alpha)=P_2\left(1+\frac{\alpha}{ik}\right)+C_1(\alpha)\,P_2\left(1+\frac{\alpha}{ik}\right)+C_2(\alpha)Q_2\left(1+\frac{\alpha}{ik}\right),$$

where $P_2$ and $Q_2$ are respectively Legendre functions of 1-st and 2-nd kind, and $C_1$ and $C_2$ the following solution:

$$\begin{cases} C_1(\alpha)=-\int_0^{\alpha} Q_2\left(1+\frac{\beta}{ik}\right)F'(\beta)\,d\beta\,, \\ C_2(\alpha)=\int_0^{\alpha} P_2\left(1+\frac{\beta}{ik}\right)F'(\beta)\,d\beta\,, \end{cases}$$

of the system

$$\begin{cases} C_1' P_2 + C_2' Q_2 = 0 , \\ C_1' P_2' + C_2' Q_2' = \dfrac{F'}{\alpha(\alpha + 2ik)} . \end{cases}$$

Then, integrating (15) by parts and differentiating we get

$$(16) \qquad \sigma_{1,k}(\alpha) = y'(\alpha) = P_2' + (P_2 Q_2' - Q_2 P_2') F(\alpha) + \int_0^{\alpha} K(\alpha, \beta) F(\beta) \, \mathrm{d}\beta ,$$

with

$$K(\alpha, \beta) = P_2'(\alpha) Q_2'(\beta) - Q_2'(\alpha) P_2'(\beta) ,$$
$$K(\alpha, \beta) = 0 \qquad \text{for} \qquad \alpha < \beta .$$

Inserting in (16) the expressions for $F$ and $P_2 Q_2' - Q_2 P_2'$ and inverting the order of the double integrals:

$$(17) \qquad \sigma_{1,k}(\alpha) = P_2' \left(1 + \frac{\alpha}{ik}\right) + \frac{1}{\alpha(\alpha + 2ik)} [C_M(\alpha) + C_T(\alpha)] +$$
$$+ \frac{1}{\alpha(\alpha+2ik)} \int_0^{\alpha-\mu} \mathrm{d}\beta [C_M(\alpha-\beta)\sigma_{1,k}(\beta) + C_T(\alpha-\beta)\varrho_{1,k}(\beta)] + \int_{\mu}^{\alpha} \mathrm{d}\beta K(\alpha, \beta)[C_M(\beta) + C_T(\beta)] +$$
$$+ \int_0^{\alpha-\mu} \mathrm{d}\gamma \int_{\gamma+\mu}^{\alpha} \mathrm{d}\beta K(\alpha, \beta)[C_M(\beta - \gamma)\sigma_{1,k}(\gamma) + C_T(\beta - \gamma)\varrho_{1,k}(\gamma)] .$$

So, the system (13) is equivalent to the system obtained by adding to the latter equation the following one

$$\alpha(\alpha + 2ik)\varrho_{1,k}(\alpha) = C_\sigma(\alpha) + C_T(\alpha) + \int_0^{\alpha-\mu} \mathrm{d}\beta [C_\sigma(\alpha - \beta)\varrho_{1,k}(\beta) + C_T(\alpha - \beta)\sigma_{1,k}(\beta)] .$$

By inspection, one sees immediately that this system of integral equations may be solved by iteration: when $\varrho_{1,k}(\alpha)$ and $\sigma_{1,k}(\alpha)$ are known for $\alpha < n\mu$ one can get them for $\alpha < (n+1)\mu$ by inserting the known functions in the right hand sides.

For instance,

$$\varrho_{1,k}(\alpha) = 0$$
$$\sigma_{1,k}(\alpha) = P' \left(1 + \frac{\alpha}{ik}\right) \qquad \text{for } \alpha < \mu \text{ etc. .}$$

Very similarly, by introducing $z(\alpha) = -1 + \int_0^\alpha \sigma_{2,k}(\beta)\,\mathrm{d}\beta$ the second equation of (13′) may be brought into the form:

$$(17') \qquad \sigma_{2,k}(\alpha) = P_2'\left(1 + \frac{\alpha}{ik}\right) - \frac{1}{\alpha(\alpha+2ik)}\,[C_M(\alpha) + C_T(\alpha)] +$$
$$+ \frac{1}{\alpha(\alpha+2ik)} \int_0^{\alpha-\mu} \mathrm{d}\beta\,[C_M(\alpha-\beta)\sigma_{2,k}(\beta) + C_T(\alpha-\beta)\varrho_{2,k}(\beta)] - \int_\mu^\alpha \mathrm{d}\beta K(\alpha,\beta)\,[C_M(\beta) - C_T(\beta)] +$$
$$+ \int_0^{\alpha-\mu} \mathrm{d}\gamma \int_{\gamma+\mu}^{\alpha} \mathrm{d}\beta K(\alpha,\beta)\,[C_M(\beta-\gamma)\sigma_{2,k}(\gamma) + C_T(\beta-\gamma)\varrho_{2,k}(\gamma)]\,,$$

which has the same structure as (17).

The general case $J \neq 1$ ($^3P_2$ and $^3F_2$, etc.) is slightly complicated by the need of transforming both integral equations of each system (13), (13′) into the form (17), (17′), but the generalization is straightforward and one gets the system

$$(18) \qquad \left.\begin{matrix} \varrho_{\lambda,k}(\alpha) \\ \sigma_{\lambda,k}(\alpha) \end{matrix}\right\} = \left\{\begin{matrix} P_{J-1}'\left(1+\frac{\alpha}{ik}\right) \\ P_{J+1}'\left(1+\frac{\alpha}{ik}\right) \end{matrix}\right. + \frac{1}{\alpha(\alpha+2ik)} \left\{\begin{matrix} C_M^{\varrho}(\alpha) + C_T(\alpha)\,, \\ [C_M^{\sigma}(\alpha) - C_T(\alpha)](\delta_{\lambda 1} - \delta_{\lambda 2}) \end{matrix}\right. +$$
$$+ \frac{1}{\alpha(\alpha+2ik)} \int_0^{\alpha-\mu} \mathrm{d}\beta \left\{\begin{matrix} C_M^{\varrho}(\alpha-\beta)\varrho_{\lambda,k}(\beta) + C_T(\alpha-\beta)\sigma_{\lambda,k}(\beta) \\ C_M^{\sigma}(\alpha-\beta)\sigma_{\lambda,k}(\beta) + C_T(\alpha-\beta)\varrho_{\lambda,k}(\beta) \end{matrix}\right. +$$
$$+ \int_\mu^\alpha \mathrm{d}\beta \left\{\begin{matrix} K_{J-1}(\alpha,\beta)[C_M^{\varrho}(\beta) + C_T(\beta)] \\ K_{J+1}(\alpha,\beta)[C_M^{\sigma}(\beta) + C_T(\beta)](\delta_{\lambda 1} - \delta_{\lambda 2}) \end{matrix}\right. +$$
$$+ \int_0^{\alpha-\mu} \mathrm{d}\gamma \int_{\gamma+\mu}^{\alpha} \mathrm{d}\beta \left\{\begin{matrix} K_{J-1}(\alpha,\beta)[C_M^{\varrho}(\beta-\gamma)\varrho_{\lambda,k}(\gamma) + C_T(\beta-\gamma)\sigma_{\lambda,k}(\gamma)] \\ K_{J+1}(\alpha,\beta)[C_M^{\sigma}(\beta-\gamma)\sigma_{\lambda,k}(\gamma) + C_T(\beta-\gamma)\varrho_{\lambda,k}(\gamma)] \end{matrix}\right.,$$

where $C_M^{\varrho,\sigma}$ and $C_T$ are defined by

$$U_M^{\varrho}(r) = \int_\mu^{+\infty} C_M^{\varrho}(\alpha)\exp[-\alpha r]\,\mathrm{d}\alpha\,,$$

$$U_M^{\sigma}(r) = \int_\mu^{+\infty} C_M^{\sigma}(\alpha)\exp[-\alpha r]\,\mathrm{d}\alpha\,,$$

$$U_T(r) = \int_\mu^{+\infty} C_T(\alpha)\exp[-\alpha r]\,\mathrm{d}\alpha\,,$$

and

$$K_L(\alpha, \beta) = P'_L(\alpha) Q'_L(\beta) - Q'_L(\alpha) P'_L(\beta) .$$

The iteration method may still be applied for solving (18).

## 5. – Analytic properties of the radial wave functions and of the $S$-matrix.

It is readily seen from the system (18) that $\varrho_{\lambda,k}(\alpha)$ and $\sigma_{\lambda,k}(\alpha)$ are well defined quantities in the $k$-plane cut along the imaginary axis from $k = i(\mu/2)$ to $i\infty$; due to the fact that the $|C'_{\sigma,M,T}(\alpha)|$ are assumed to be bound by $M\alpha^{1-\varepsilon}$, one may show that $f_\lambda(r, k)$, $g_\lambda(r, k)$, $f_\lambda(k)$ and $g_\lambda(k)$ are analytic functions in the $k$-plane cut from $+ i(\mu/2)$ to $+ i\infty$, by using the same kind of arguments as those presented in ref. (1).

According to the eq. (7), (8) and (8′), the elements of the $S$-matrix in the *JLS* scheme (*) have the same analytic properties as $f_\lambda(k)$ and $g_\lambda(k)$ except additional poles introduced by algebraic operations, actually these poles occur when

$$\frac{f_1(+k)}{g_1(-k)} = \frac{f_2(-k)}{g_2(-k)} , \tag{19}$$

and, in the upper half $k$-plane, they correspond to bound states: indeed, when (19) holds, from (8) and (8′) one must have $c = c' = 0$, so that a *regular* solution of (3) is

$$\begin{pmatrix} u(r) \\ w(r) \end{pmatrix} = a_1 \psi_1^+ + a_2 \psi_2^+ \underset{r\to\infty}{\simeq} \begin{pmatrix} (a_1 + a_2) \exp[ikr] \\ (a_1 - a_2) \exp[ikr] \end{pmatrix} . \tag{20}$$

Now, due to the reality of the potentials, one can deduce very easily from (3) the following relation:

$$(u^* u' - u u'^* + w^* w' - w w'^*) \Big|_0^r + (k^2 - k^{*2}) \int_0^r (|u|^2 + |w|^2) \, \mathrm{d}r' = 0 ,$$

which, in the present case, writes

$$(k + k^*) \left[ 2i(|a_1|^2 + |a_2|^2) + (k - k^*) \int_0^{+\infty} (|u|^2 + |w|^2) \, \mathrm{d}r \right] = 0 ,$$

(*) It is not convenient to study the analytic properties of the eigenphase-shifts because they are artificially complicated by the fact that $\delta^{\alpha,\gamma}(k)$ has two determinations.

this shows that, for $\mathrm{Im}\, k > 0$, $\mathrm{Re}\, k = 0$, and the above mentioned result follows.

In the very special case when *either* the system (8) *or* (8′) is undetermined, it is easily seen (from the symmetry property of the $S$-matrix) that *both* systems must be undetermined: then, this means that the regular solution is a 3-parameter-dependent solution and it would behave as an usual 2-parameter-dependent regular solution only for $k = 0$.

## 6. – Conclusions.

We have given a method of calculation of phase shifts and admixture parameters directly from the inverse Laplace transform of the potential in the case of non central forces; this is of some interest because it often happens that a potential is given in terms of its inverse Laplace transform, for instance when it is derived from field-theoretical considerations [1]. This method may look somewhat complicated for practical purposes, but there is no really simple method of solution of the Schrödinger equation when tensor forces are present.

At the same time, we have shown that the $S$-matrix has, in the present case, the same analytic features as in the case of central forces.

Finally, we obtain, as a by-product, integral representations of the wave functions which might be useful to study the analytic properties of non-relativistic matrix elements.

## Appendix

We wish to prove here the relation:

$$f_1(r,k)g_1(r,-k) - f_1(r,-k)g_1(r,k) + f_2(r,k)g_2(r,-k) - f_2(r,-k)g_2(r,k) = 0\,, \tag{A.1}$$

which has been used several times.

Clearly, this relation is equivalent to:

$$u_1^+ w_1^- - u_1^- w_1^+ + u_2^+ w_2^- - u_2^- w_2^+ = 0\,, \tag{A.2}$$

where we have put, for sake of brevity

$$\begin{pmatrix} u_\lambda^+ \\ w_\lambda^+ \end{pmatrix} = \begin{pmatrix} f_\lambda(r,-k)\exp[ikr] \\ g_\lambda(r,-k)\exp[ikr] \end{pmatrix}, \qquad \begin{pmatrix} u_\lambda^- \\ w_\lambda^- \end{pmatrix} = \begin{pmatrix} f_\lambda(r,k)\exp[-ikr] \\ g_\lambda(r,k)\exp[-ikr] \end{pmatrix}, \qquad (\lambda = 1,2)\,.$$

[1] J. M. Charap and S. Fubini: *Nuovo Cimento*, **14**, 540 (1959); **15**, 73 (1960); J. M. Charap and M. J. Tausner: *Nuovo Cimento*, **18**, 316 (1960).

Now, from (3) we may deduce:

$$(u_2^+ u_1^{+\prime} - u_1^+ u_2^{+\prime} + w_2^+ w_1^{+\prime} - w_1^+ w_2^{+\prime})\,|_r^\infty = 0\,,$$

*i.e.*

$$u_2^+ u_1^{+\prime} - u_1^+ u_2^{+\prime} + w_2^+ w_1^{+\prime} - w_1^+ w_2^{+\prime} = 0\,, \tag{A.3}$$

and in the same way, the following relations may be obtained

$$u_2^- u_1^{+\prime} - u_1^+ u_2^{-\prime} + w_2^- w_1^{+\prime} - w_1^+ w_2^{-\prime} = 0\,, \tag{A.4}$$

$$u_2^- u_1^{-\prime} - u_1^- u_2^{-\prime} + w_2^- w_1^{-\prime} - w_1^- w_2^{-\prime} = 0\,, \tag{A.5}$$

$$u_2^+ u_1^{-\prime} - u_1^- u_2^{+\prime} + w_2^+ w_1^{-\prime} - w_1^- w_2^{+\prime} = 0\,. \tag{A.6}$$

Multiplying (A.3), (A.4), (A.5) and (A.6) respectively by $u_2^- u_1^-$, $-u_2^+ u_1$, $u_2^+ u_1^+$ and $-u_2^- u_1^+$ and adding them together we get

$$(u_2^- w_2^+ - u_2^+ w_2^-)(u_1^- w_1^{+\prime} - u_1^+ w_1^{-\prime}) - (u_1^- w_1^+ - u_1^+ w_1^-)(u_2^- u_2^{+\prime} - u_2^+ w_2^{-\prime}) = 0\,. \tag{A.7}$$

Furthermore, noticing that the relations (A.3), (A.4), (A.5) and (A.6) are invariant by $u_\lambda^+ \leftrightarrow w_\lambda^+$ and $u_\lambda^- \leftrightarrow w_\lambda^-$ we may obtain

$$(u_2^- w_2^+ - u_2^+ w_2^-)(w_1^- u_1^{+\prime} - w_1^+ u_1^{-\prime}) - (u_1^- w_1^+ - u_1^+ w_1^-)(w_2^- u_2^{+\prime} - w_2^+ u_2^{-\prime}) = 0\,. \tag{A.8}$$

Substracting (A.8) from (A.7) and then integrating we get

$$\begin{vmatrix} u_1^+ w_1^- - u_1^- w_1^+ \\ u_2^+ w_2^- - u_2^- w_2^+ \end{vmatrix} = c\,, \tag{A.9}$$

the constant $c$ is readily determined by (A.7) or (A.8) for $r \to \infty$: actually $c = -1$ and the relation (A.2) is proved.

***Note added in proof.***

It was pointed out to us by Dr. H. Cornille from Orsay who worked independently on the same subject that the behaviour

$$f_\lambda(r, k) \simeq \alpha r^{1-J}\,,$$

$$g_\lambda(r, k) \simeq \beta r^{-1-J}\,,$$

near the origin is correct only if the tensor potential $V_T(r)$ vanishes at the origin. If, near $r = 0$, $V_T(r) \simeq C r^{-\varepsilon}$ $(0 < \varepsilon < 2)$ then

$$f_\lambda(r, k) \simeq \alpha r^{1-J-\varepsilon}\,,$$

$$g_\lambda(r, k) \simeq \beta r^{-1-J}\,,$$

if $\varepsilon = 0$

$$f_\lambda(r, k) \simeq \alpha r^{1+J} \log r ,$$

$$g_\lambda(r, k) \simeq \beta r^{-1-J} .$$

For the construction of the $S$ matrix the quantities $f_\lambda(k)$, $g_\lambda(k)$ have to be redefined correspondingly.

## RIASSUNTO (*)

Precedenti lavori hanno mostrato le proprietà notevolmente semplici dell'ampiezza parziale dell'onda di scattering e delle funzioni radiali corrispondenti derivate da un potenziale, che è una sovrapposizione continua dei potenziali esponenziali o di Yukawa. Noi estendiamo questi risultati al caso in cui nello scattering di due particelle con spin $\frac{1}{2}$ sono presenti forze non centrali. Esponiamo un metodo per calcolare gli elementi della matrice $S$ in termini delle trasformazioni di Laplace inverse dei vari potenziali radiali, formuliamo le proprietà analitiche di questi elementi di matrice ed otteniamo rappresentazioni integrali delle funzioni d'onda.

(*) *Traduzione a cura della Redazione.*

# The Effect of the Exclusion Principle on the Real Part of the Optical Potential.

J. Dabrowski

*Institute of Nuclear Research,*
*Institute for Theoretical Physics, Warsaw University - Warsaw*

A. Sobiczewski

*Institute for Theoretical Physics, Warsaw University,*
*Warsaw Technical University - Warsaw*

(ricevuto il 16 Novembre 1960)

The present paper deals with the derivation of the real part of the optical potential for the nucleon-nucleus scattering, based on our knowledge of the nuclear forces. By taking into account the corrections that result from the exclusion principle we hope to improve this derivation.

The general theory of the optical potential has been given by Watson and his collaborators ([1-3]). The set of basic equations of that theory has been reviewed, *e.g.*, in ref. ([4-6]). In particular in ref. ([6]) an interation procedure has been developed, which enables us to express the $t$-matrix for the nucleon-nucleon scattering in the nuclear medium with the help of the free nucleon-nucleon scattering matrix $t_0$. For the case $M^* = M$ we get as the first approximation of the real part of the $t$-matrix:

$$\text{Re}\, t_1(k)_K = \text{Re}\, \{t_0(k) + \Delta(k)_K + \Delta'(k)_K\}\,, \tag{1}$$

where $t_0(k) = \langle \boldsymbol{k} | t_0 | \boldsymbol{k} \rangle$, $t_1(k)_K = \langle \boldsymbol{k} | t_1 | \boldsymbol{k} \rangle_K$ are the diagonal elements of the $t_0$, $t_1$ matrices in the momentum representation ($\hbar k$ and $\hbar K$ are the relative and the center of mass

---

([1]) K. M. Watson: *Phys. Rev.*, **89**, 575 (1953).
([2]) N. C. Francis and K. M. Watson: *Phys. Rev.*, **92**, 291 (1953).
([3]) G. Takeda and K. M. Watson: *Phys. Rev.*, **97**, 1336 (1955).
([4]) J. Dąbrowski and J. Sawicki: *Nucl. Phys.*, **13**, 621 (1959).
([5]) J. Dąbrowski and A. Sobiczewski: *Acta Phys. Pol.*, in press.
([6]) J. Dąbrowski and J. Sawicki: *Nucl. Phys.*, **22**, 318 (1961).

momenta respectively), and

$$\text{(2)}\qquad \mathrm{Re}\,\Delta(k)_K = \frac{\pi k M}{2\hbar^2}\int \mathrm{d}\Omega_{k''}\left[Q\left(\frac{1}{2}\boldsymbol{K}+\frac{\boldsymbol{k}''}{k''}k\right)-1\right]\mathrm{Im}\left\{\langle \boldsymbol{k}|t_0|\frac{\boldsymbol{k}''}{k''}k\rangle\langle\frac{\boldsymbol{k}''}{k''}k|t_0|\boldsymbol{k}\rangle\right\},$$

$$\text{(3)}\qquad \mathrm{Re}\,\Delta'(k)_K = \frac{M}{\hbar^2}\mathscr{P}\int\frac{\mathrm{d}\boldsymbol{k}''}{k^2-k''^2}\left[Q\left(\frac{1}{2}\boldsymbol{K}\pm\boldsymbol{k}''\right)-1\right]\mathrm{Re}\,\{\langle\boldsymbol{k}|t_0|\boldsymbol{k}''\rangle\langle\boldsymbol{k}''|t_0|\boldsymbol{k}\rangle\},$$

where

$$\text{(4)}\qquad Q(\tfrac{1}{2}\boldsymbol{K}\pm\boldsymbol{k}'') = \begin{cases} 1 \text{ for } |\tfrac{1}{2}\boldsymbol{K}\pm\boldsymbol{k}''|>k_F, \\ 0 \text{ otherwise}. \end{cases}$$

The $\Delta'$ part is difficult to calculate since it requires knowledge of the off-energy-shell elements of the $t_0$ matrix. For this reason, and since at higher energies the on-energy-shell elements are likely to be of more importance, we will neglect $\Delta'(k)_K$. This approximation seems to be justified, since in a model example considered later the terms $\mathrm{Re}\,\{t_0+\Delta\}$ reproduce the essential part of $\mathrm{Re}\,t$.

Furthermore, we approximate

$$\left\{\langle\boldsymbol{k}|t_0|\frac{\boldsymbol{k}''}{k''}k\rangle\langle\frac{\boldsymbol{k}''}{k''}k|t_0|\boldsymbol{k}\rangle\right\}$$

in eq. (1) by its isotropic part $\{\}_{is}$ and obtain

$$\text{(5)}\qquad \mathrm{Re}\,\Delta(k)_K \simeq \frac{\pi k M}{2\hbar^2}\mathrm{Im}\left\{\langle\boldsymbol{k}|t_0|\frac{\boldsymbol{k}''}{k''}k\rangle\langle\frac{\boldsymbol{k}''}{k''}k|t_0|\boldsymbol{k}\rangle\right\}_{is}\int\mathrm{d}\Omega_{k''}\left[Q\left(\frac{1}{2}\boldsymbol{K}\pm\frac{\boldsymbol{k}''}{k''}k\right)-1\right].$$

By this isotropic approximation $\{\}\simeq\{\}_{is}$ we neglect all the terms proportional to $P_2(\cos\theta)$, $P_4(\cos\theta)$, ... ($\theta$ is the scattering angle) in the expansion $\{\}=\sum A_L P_L$, because the $\Omega_{k''}$ integration leaves the even $L$ terms only. Since, however, the factor $Q-1$ anyhow severely restricts the $\Omega_{k''}$ integration region, our approximation seems to be justified.

Both the terms $\mathrm{Re}\,t_0$ and $\mathrm{Re}\,\Delta$ have to be averaged over the spin and iso-spin states of the nucleons. The averaged values $\mathrm{Re}\,\tilde{t}_0$ and $\mathrm{Re}\,\tilde{\Delta}$ may be expressed easily by the nucleon-nucleon phase shifts. The result is:

$$\text{(6)}\qquad \mathrm{Re}\,\tilde{t}_0(k) = -\frac{\hbar^2}{32\pi^2 M}\frac{1}{k}g(k),$$

$$\text{(7)}\qquad \mathrm{Re}\,\tilde{\Delta}(k)_K = \frac{\hbar^2}{32\pi^2 M}\frac{2}{k}g'(k)\left\{\int\frac{\mathrm{d}\Omega_{k''}}{4\pi}Q\left(\frac{1}{2}\boldsymbol{K}\pm\frac{\boldsymbol{k}''}{k''}k\right)-1\right\},$$

where

$$\text{(8)}\qquad g(k) = \Big[\sum_{\text{odd } j}+3\sum_{\text{even } j}\Big](2j+1)\sin 2\delta_j + \Big[\sum_{\text{even } j}+3\sum_{\text{odd } j}\Big]\big[(2j+1)\sin 2\eta_j + (2j+3)(\sin 2\eta_{j+1\,\alpha}+\mathrm{sen}\,2\eta_{j+1\,\beta})\big]+3\sin 2\eta_0,$$

$$(9)\quad g'(k) = \Big[\sum_{\text{odd}\,j} + 3\sum_{\text{even}\,j}\Big](2j+1)\,\sin 2\delta_j \sin^2\delta_j + \Big[\sum_{\text{even}\,j} + 3\sum_{\text{odd}\,j}\Big]\cdot$$
$$\cdot\big[(2j+1)\sin 2\eta_j \sin^2\eta_j + (2j+3)(\sin 2\eta_{j+1\alpha}\sin^2\eta_{j+1\alpha} +$$
$$+ \sin 2\eta_{j+1\beta}\sin^2\eta_{j+1\beta})\big] + 3\sin 2\eta_0 \sin^2\eta_0\,,$$

where $\delta_j$ denotes the singlet phase shift and $\eta_j$, $\eta_{j\alpha}$, $\eta_{j\beta}$ denote the uncoupled ($l=j$) and respectively coupled ($l=j-1$, $l=j+1$) triplet phase shifts of Blatt and Biedenharn.

With our approximated $\mathrm{Re}\,\tilde{t}_1 \simeq \mathrm{Re}\,\tilde{t}_0 + \mathrm{Re}\,\tilde{\Delta}$ we may now (see ref. [1-6]) calculate the corresponding approximation of the real part of the optical potential, $\mathrm{Re}\,\mathscr{V}_1$, which has the form

$$(10)\qquad \mathrm{Re}\,\mathscr{V}_1 = \mathrm{Re}\,\mathscr{V}_0 + \mathrm{Re}\,\Delta\mathscr{V}\,,$$

where $\mathrm{Re}\,\Delta\mathscr{V}$ is the correction term due to the Pauli principle, arising from the $\Delta$ term of eq. (1).

Though all our approximations seem to be reasonable, especially at higher energies, we have not succeeded in estimating the error introduced by these approximations. For this reason we are going to investigate a model example in which we know the « exact » scattering matrix $t$. Our model example is the case of the separable Yamaguchi interaction [7]. For this case the calculation of the « exact » optical potential $\mathscr{V}$ was performed by VERLET and GAVORET [8] who in particular calculated both $\mathrm{Re}\,\mathscr{V}$ and $\mathrm{Re}\,\mathscr{V}_0$. With the help of the same values of all the parameters as in ref. [8] we have calculated $\mathrm{Re}\,\Delta\mathscr{V}$. The results are shown in Fig. 1. The curves $\mathrm{Re}\,\mathscr{V}$ and $\mathrm{Re}\,\mathscr{V}_0$ have been drawn by using the results of ref. [8].

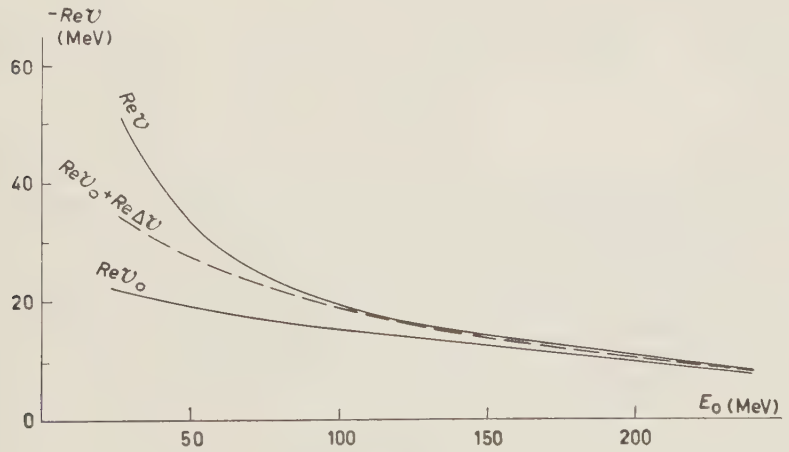


Fig. 1.

Watson's theory contains approximations which restrict its application to the incoming nucleon energies $E_0 \gtrsim (50 \div 100)$ MeV. We notice that at these energies

(7) Y. YAMAGUCHI: *Phys. Rev.*, **95**, 1628 (1954).
(8) L. VERLET and J. GAVORET: *Nuovo Cimento*, **10**, 505 (1958).

the curve $\mathrm{Re}\,\mathscr{V}_0+\mathrm{Re}\,\Delta\mathscr{V}$ is very close to the « exact » $\mathrm{Re}\,\mathscr{V}$ curve. We conclude then that our procedure turns out — at least in our model example — to render a good approximation to the « exact » $\mathrm{Re}\,\mathscr{V}$.

Another feature of the resuts shown in Fig. 1 is that the Pauli principle is effective even at comparatively high energies, changing essentialy the real part of the optical potential at, let us say, $E_0 \sim 100$ MeV.

We now are going to present the results obtained for $\mathrm{Re}\,\mathscr{V}_0+\mathrm{Re}\,\Delta\mathscr{V}$ with the help of the Signell-Marshak phase shifts ([9]). We assumed the nuclear density $\varrho_0$ to correspond to the value of $1.27f$ for $r_0=(\frac{4}{3}\pi\varrho_0)^{-\frac{1}{3}}$ ($k_F=1.2f^{-1}$) and the energy of the top of the Fermi sea — 10 MeV. We performed the calculation for 3 energies $E_0 \simeq 30$, 80 and 230 MeV. The results are shown in Fig. 2. Fig. 2 also contains the experimental points taken from ref. ([10]).

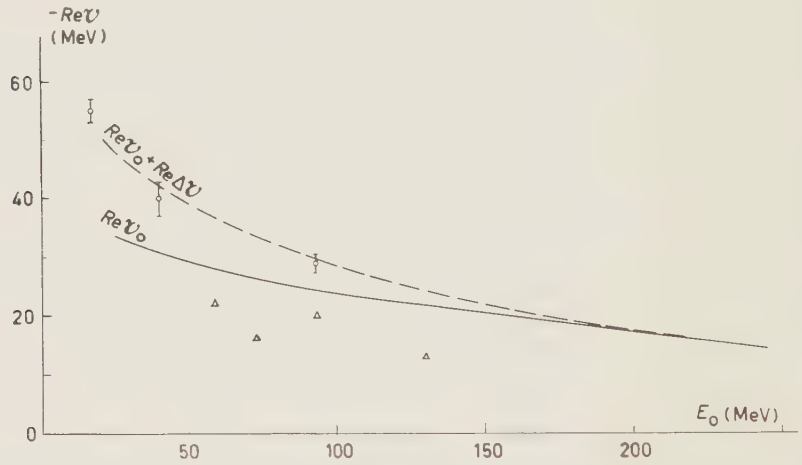


Fig. 2.

The first thing we notice is an essential increase in the real part of the optical potential at medium energies, caused by the exclusion principle. At high energies ($\sim 200$ MeV) the exclusion principle has practically no effect on $\mathrm{Re}\,\mathscr{V}$. Let us notice that the situation with $\mathrm{Re}\,\mathscr{V}$ is exactly opposite to the one with $\mathrm{Im}\,\mathscr{V}$. Namely, the exclusion principle diminishes the depth of the imaginary potential. This, in particular, most probably results in a surface absorption (see, *e.g.*, ([8,11])). In contradistinction to this situation with $\mathrm{Im}\,\mathscr{V}$ our results for $\mathrm{Re}\,\mathscr{V}$ would indicate that one should expect the real potential to be concentrated more in the middle of the nucleus, where the Pauli principle is more effective and the potential is correspondingly deeper (*).

Comparison with experiment is difficult, because of the uncertainty of the experimental results shown in Fig. 2. Moreover, the experimental points give the

([9]) P. S. Signell and R. E. Marshak: *Phys. Rev.*, **109**, 1229 (1958).
([10]) A. E. Glassgold: *Progr. Nucl. Phys.*, **7**, 124 (1959).
([11]) G. L. Shaw: *Ann. Phys.*, **8**, 509 (1959).
(*) A calculation of $\mathrm{Re}\,\mathscr{V}_0+\mathrm{Re}\,\Delta\mathscr{V}$ for a finite nucleus in the Thomas-Fermi approximation is now in progress.

parameters of the central part of the potential, the radial dependence of which was assumed to be of the form $[1+\exp\{(r-R)/a\}]^{-1}$. There might be some doubts whether one can compare these experimental results with the results of our calculation for an infinite nuclear medium. With all these reservations in mind we notice a reasonabe agreement of our Re $\mathscr{V}_0$ + Re $\Delta\mathscr{V}$ curve with the experimental results of GLASSGOLD *et al.* [12], which in Fig. 2 are indicated by circles. The fact that — in contradistinction to the known experimental results — Re $\mathscr{V}_0$ does not vanish at $E_0 \sim 300$ MeV, obviously cannot be changed by our Re $\Delta\mathscr{V}$ correction, which practically vanishes at such high energies.

It is interesting to look for some intuitive explanation of the increase in Re $\mathscr{V}$ resulting from the Pauli principle. One may understand the whole effect in the following way (*). The situation is analogouos to the chemical binding effect in the low energy nucleon-nucleon scattering, where because of the binding to a molecule the nucleon is more resistant to the force exercised on it by the scattered nucleon. This binding effect increases the nucleon-nucleon scattering cross section, or one may say that because of the binding the effective nucleon-nucleon interaction increases. Similarly, in our problem nucleons that are inside of the Fermi sea, because of the Pauli principle, cannot undergo small momentum changes. This increases their resistance to the force exercised on them by the projectile. As a result we expect an increase in the effective interaction.

---

[12] A. E. GLASSGOLD and P. J. KELLOGG: *Phys. Rev.*, **109**, 1291 (1958).

(*) This explanation was suggested to the authors by Y. YAMAGUCHI.

IL NUOVO CIMENTO Vol. XX, N. 2 16 Aprile 1961

# Effects of the $\gamma$-3$\pi$ Interaction on Photoreactions.

M. Kawaguchi and Y. Miyamoto

*Physics Department, Tokyo University of Education - Tokyo*

Y. Fujii

*Physics Department, College of Science and Engineering, Nihon University - Tokyo*

(ricevuto il 2 Febbraio 1961)

**1.** – Recent investigations show that the $P$-wave pion-pion interaction plays an important role in pion processes [1,3]. It seems to be possible, at least qualitatively, to interpret the isovector part of the nucleon electromagnetic form factors, the spin-orbit potential of nuclear forces [4], the higher resonances of pion-nucleon scattering and pion photoproduction [1], in terms of the pion-pion resonance in the $I=J=1$ state. Two of us (Y.F. and Y.M.) have discussed these processes in the Mandelstam representation and the chain approximation.

**2.** – In this situation, we are naturally led to examine the effects of the $\gamma$-3$\pi$ interaction in photoreactions. In the present note, it is pointed out that the $\gamma$-3$\pi$ interaction, which is responsible for the isoscalar part of the electromagnetic form factors of a nucleon [5], is also important for pion photoproduction (Fig. 1*a*) [6,7], the magnetic moment of the deuteron, and the photodisintegration of the deuteron (Fig. 1*b*). Since we do not have any direct way of finding the effective coupling constant of the $\gamma$-3$\pi$ interaction, discussion is focussed on obtaining information about the sign and the magnitude of the coupling constant from various processes.

(1) K. Itabashi, M. Kato, K. Nakagawa and G. Takeda: *Progr. Theor. Phys.*, **24**, 529 (1960).
(2) Y. Miyamoto: *Progr. Theor. Phys.*, **24**, 840 (1960).
(3) W. R. Frazer and J. R. Fulco: *Phys. Rev.*, **117**, 1609 (1960).
(4) Y. Fujii: *Progr. Theor. Phys.*, **25**, 441 (1961).
(5) P. Federbush, M. L. Goldberger and S. B. Treiman: *Phys. Rev.*, **112**, 642 (1958).
(6) J. S. Ball: *Phys. Rev. Lett.*, **5**, 73 (1960).
(7) M. Kato: *Progr. Theor. Phys.*, **25**, 493 (1961); B. de Tollis, E. Ferrari and H. Munczek: *Nuovo Cimento*, **18**, 198 (1960).

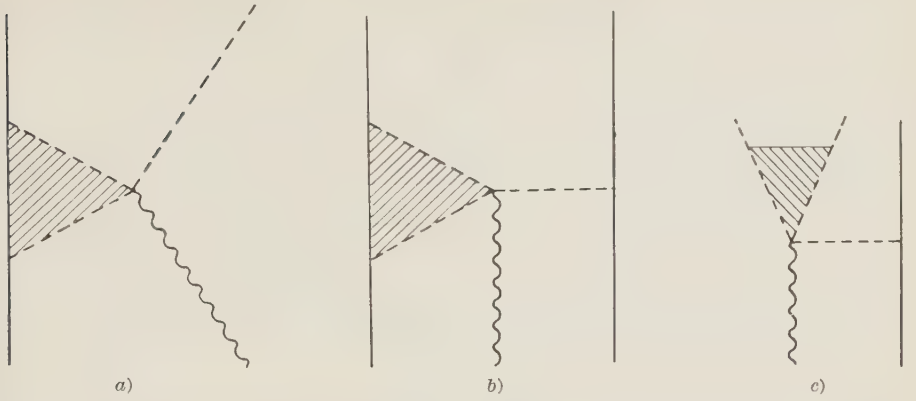


Fig. 1. – Diagrams of photoreactions due to the $\gamma$-3$\pi$ interaction. The nucleon is expressed by a solid line, the pion by a broken line, and the photon by a wavy line: *a*) is the diagram of pion photoproduction; *b*) is for the magnetic and electric quadrupole moments of the deuteron, and also for the photodisintegration; and *c*) for the double pion photoproduction. The shaded part is replaced by the isovector part of the electromagnetic form factors.

**3.** – The effective Hamiltonian is given by [8]

$$H = \frac{ie\lambda}{\mu^3}\,\varepsilon_{\lambda\mu\nu\varrho}\,A_\lambda\,\frac{\partial\varphi_1}{\partial x_\mu}\,\frac{\partial\varphi_2}{\partial x_\nu}\,\frac{\partial\varphi_3}{\partial x_\varrho}\,, \tag{1}$$

where $\lambda$ is the effective coupling constant, and $\mu$ is the pion mass. From a nucleon closed loop graph in lowest order perturbation approximation, we obtain a value for $\lambda$,

$$\lambda_0 = \frac{16}{\sqrt{\pi}}\,f^3 \approx 0.2\,. \qquad \text{for } f^2 = 0.08: \tag{2}$$

This is of the right sign, but seems to be too small to explain the isoscalar part of the nucleon charge distribution, according to Hiida-Nakanishi's detailed calculation without pion-pion interaction [9].

Since any pair of pions in the Hamiltonian (1) is in the state with $I=J=1$, the enhanced pion-pion interaction must be taken into account. The coupling constant $\lambda_0$ is, therefore, modified to $\lambda_1$. For definiteness we shall discuss this effect on the basis of the chain approximation method [2]. If we neglect the pion-pion re-scattering correction completely in the diagrams in Fig. 1*a* and Fig. 1*b*, we have a factor

$$M_0 = \lambda_0 G_\alpha(q^2)\,, \qquad (\alpha = 1, 2), \tag{3}$$

where $G_\alpha(q^2)$ is the isovector part of the nucleon electro-magnetic form factor without the pion-pion interaction. If we substitute the chain diagram, illustrated

---

[8] B. Bosco and V. de Alfaro: *Phys. Rev.*, **115**, 215 (1959).
[9] K. Hiida and N. Nakanishi: *Progr. Theor. Phys.*, **22**, 863 (1959).

in Fig. 2, for the pion-pion scattering part in the shaded area of Fig. 1*a* and Fig. 1*b*, $M_0$ is altered in the following way,

Fig. 2. – Pion-pion interaction in the chain approximation.

$$M_0 \to M = \frac{M_0}{1 - X(q^2)}, \tag{4}$$

where $X(q^2)$ is the contribution from a single bubble in Fig. 2, and is given by eq. (3.4) of ref. (2). In terms of the modified coupling constant $\lambda_1$ and the « observed » electromagnetic form fator $G_\alpha(q^2)$, which includes the pion-pion re-scattering correction, $M$ can be written as

$$M = \lambda_1 G_\alpha(q^2) . \tag{5}$$

Here $\lambda_1$ and $G_\alpha(q^2)$ are given by

$$\begin{cases} \lambda_1 = \lambda_0 \dfrac{a}{a_0} \dfrac{1}{D(0)} , \\ G_\alpha(q^2) = G_\alpha(q^2) F(q^2) = G_\alpha(q^2) \dfrac{D(0)}{D(q^2)} , \end{cases} \tag{6}$$

where $a_0$ and $a$ are, respectively, the bare and the renormalized pion-pion scattering lengths in the state $I = J = 1$, and $D(q^2)$ is the denominator appearing in the pion-pion scattering amplitude given by eq. (2.5′) and (4.3) in ref. (2) (*), and $F(q^2)$ is the pion electromagnetic form factor.

It is obvious that $\lambda_1$ also appears in the calculation of the isoscalar parts of the nucleon electromagnetic form factor. Furthermore we can show that $\lambda_0$ is modified in the following way,

$$\lambda_0 \to \frac{\lambda_0}{1 - X(q^2)} = \lambda_1 F(q^2) , \tag{7}$$

in the $\gamma$-$2\pi$ process illustrated in Fig. 1*c*.

From the above considerations we should be interested in $\lambda_1$, which appears commonly in the various processes, rather than in $\lambda_0$. The value in eq. (2) should be considered as giving only an idea of the sign and rough magnitude of $\lambda_1$.

## 4. – Plus-minus ratio in pion photoproduction.

The additional term due to the $\gamma$-$3\pi$ interaction is given by calculating the diagram in Fig. 1*a*. The shaded part is replaced by the isovector part of the nucleon form factors $G_\alpha(q^2)$, namely

$$\left\langle \varphi_i \frac{\partial}{\partial x_\mu} \varphi_j - \frac{\partial}{\partial x_\mu} \varphi_i \varphi_j \right\rangle_0 = -\frac{i}{e} [\gamma_\mu \tau_k G_1(q^2) - \sigma_{\mu\nu} q_\nu \tau_k G_2(q^2)] , \quad (i, j, k \text{ cyclic}), \tag{8}$$

(*) $D(q^2)$ is written as a function of $q^2$ instead of $\nu$ in ref. (2).

where

$$G_1(0) = \frac{e}{2}\zeta\,, \qquad G_2(0) = \frac{e}{2m}\frac{1}{2}(\mu_p - \mu_n)\,.$$

Here $\zeta$ is the dissociation probability of nucleon, $m$ is the nucleon mass, $\mu_p$ and $\mu_n$ are the anomalous magnetic moments of the proton and neutron, respectively, and $q$ is the four-momentum transfer.

The plus-minus ratio at threshold is

$$R \equiv \frac{\sigma(-)}{\sigma(+)} = \frac{[1+\mu/2m+(\mu/2m)(\mu_p+\mu_n)+(\lambda_1/g)\{\frac{1}{4}(\mu_p-\mu_n)+\frac{1}{2}\zeta\}]^2}{[1-\mu/2m-(\mu/2m)(\mu_p+\mu_n)-(\lambda_1/g)\{\frac{1}{4}(\mu_p-\mu_n)+\frac{1}{2}\zeta\}]^2}\,, \tag{9}$$

where $g^2/4\pi = 15$ (*).

Comparing with the experimental value $R=1.35$, we obtain a positive $\lambda_1$ lying between

$$\begin{cases} \lambda_1 = 0.26 \quad \text{for} \quad \zeta = 0 \\ \text{and} \\ \lambda_1 = 0.17 \quad \text{for} \quad \zeta = 1\,. \end{cases} \tag{10}$$

This sign is compatible with the isoscalar part of the nucleon form factors. If we take the recent experimental value $R=1.08$ [10] which seems to be too small, $\lambda_1$ becomes negative, lying between

$$\begin{cases} \lambda_1 = -0.75 \quad \text{for} \quad \zeta = 0 \\ \text{and} \\ \lambda_1 = -0.48 \quad \text{for} \quad \zeta = 1\,. \end{cases} \tag{11}$$

## 5. – Magnetic and electric quadrupole moments of the deuteron.

The difference between the magnetic moment of the deuteron and of its constituents is usually interpreted in terms of the $D$-state probability, $P_D$, as

$$\begin{aligned} \mu_d - (1+\mu_p+\mu_n) &= -\tfrac{3}{2}P_D(1+\mu_p+\mu_n-\tfrac{1}{2})\,, \\ &= -0.022\,. \end{aligned} \tag{12}$$

As a part of the exchange moment, the diagram given in Fig. 1*b* contributes to the magnetic moment. It is well known that the one pion exchange moment

(*) It is noted that eq. (9) satisfies the Kroll-Ruderman theorem. The term proportional to $\lambda_1$ vanishes when $\mu$ tends to zero. Actually, the Hamiltonian (1) contains derivatives of meson fields which tend to zero. $\mu^3$ in the denominator must be fixed, because it is introduced for dimensional reason.

[10] W. P. Swanson, D. G. Gates, T. L. Jenkins and R. W. Kenney: *Phys. Rev. Lett.*, **5**, 336 (1950).

vanishes because of the isotopic spin conservation. On the other hand, we shall show that the $\gamma$-3$\pi$ effect is also responsible for the non-additivity of the magnetic moment, because it does not change the isotopic spin.

The preliminary result is

$$\mu_{\gamma\text{-}3\pi} = -0.1\lambda_1\zeta \,. \tag{13}$$

Eq. (12) is, therefore, revised to

$$-0.1\lambda_1\zeta - \tfrac{3}{2}0.379\, P_D = -0.022 \,. \tag{14}$$

Hence, the $\gamma$-3$\pi$ interaction makes the $D$-state probability smaller. It is, however, difficult to draw any definite conclusion, because there are other relativistic effects ([11]) and a contribution from the spin-orbit force ([12]).

Incidentally the effect of the $\gamma$-3$\pi$ interaction on the electric quadrupole moment is about 20% of the observed value.

## 6. – Photodisintegration of the deuteron.

The photodisintegration of the deuteron in the high energy region around $E_\gamma \approx 250$ MeV is usually described in terms of $E1$ transitions with $^3P$ final states and $M1$ transitions into a $^1D$ state through the (33) pion-nucleon resonance ([13]). Other transitions are considered to be relatively small. Apparently very small asymmetry with respect to 90° is expected for the theoretical angular distribution, while experiment shows a big forward peak.

Now the $\gamma$-3$\pi$ interaction contributes to $M1$ transition into $^3S$, and probably to $E1$ transition into $^1P$ state, since the final state must be isotopic spin singlet. Consequently, a considerable amount of $\cos\theta$ term can be expected from the interference between our terms and the usual ones.

Detailed account will be published in the *Progr. Theor. Phys.*

---

([11]) H. MIYAZAWA: *Progr. Theor. Phys.*, **7**, 207 (1952).
([12]) H. FESHBACH: *Phys. Rev.*, **107**, 1626 (1957).
([13]) T. AKIBA: *Progr. Theor. Phys.*, **24**, 370 (1960).

# The Nucleon Core in High-Energy Neutrino Processes.

N. Cabibbo

*Laboratori Nazionali del C.N.E.N. - Frascati*
*Istituto di Fisica dell'Università - Roma*

(ricevuto il 23 Febbraio 1961)

**1**. – Recent results on the elastic scattering of electrons show that the proton has a hard core in which part of its charge is concentrated. It seems reasonable to expect that this feature of the structure of the nucleons will manifest itself also in their weak interactions with high momentum transfer, in reactions of the kind:

$$(1) \qquad \begin{cases} \bar{\nu}+p \to n+e^+, \\ \nu+n \to p+e^-. \end{cases}$$

These processes have been investigated in a precedent letter [1,2]. Under the assumptions specified in I the differential cross sections for these processes were found (I, (7), (8)) to depend essentially on three form factors $F_1$, $F_2$, $H_1$, functions of the momentum transfer $k^2$. These were supposed to be connected to the electromagnetic form factors of the nucleon through the relations

$$(2) \qquad F_1(k^2) = F_1^V(k^2) ,$$

$$(3) \qquad F_2(k^2) = F_2^V(k^2) ,$$

$$(4) \qquad H_1(k^2) = \frac{G_A}{G} F_1^V(k^2) ,$$

$F_1^V$ and $F_2^V$ are the form factors for the isotopic vector part of the charge and anomalous magnetic moment of the nucleon. While (2) and (3) are consequences of the hypothesis of conservation of the vector current in β-interaction, (4) is a reasonable guess.

---

(1) N. Cabibbo and R. Gatto: *Nuovo Cimento*, **15**, 304 (1960), hereafter called I.

(2) B. Pontecorvo, *Žurn. Éksp. Teor. Fiz.*, **37**, 1751 (1959); T. D. Lee and C. L. Yang: *Phys. Rev. Lett.*, **4**, 307 (1960); Y. Yamaguchi: *Prog. Theor. Phyr.*, **23**, 1117 (1960). See also G. Bernardini: *Proceedings of the 1960 Annual International Conference on High Energy Physics at Rochester*, p.581.

Experimental evidence at that time favored an exponential distribution for the density of both charge and magnetic moment of the nucleon, giving

$$F_1^V(k^2) \approx F_2^V(k^2) \approx \left(1 + \frac{k^2}{a^2}\right)^{-2}. \tag{5}$$

The total cross-section for processes (1) thus evaluated (see I) increases at low energy with the square of the c.m. energy, but levels off, at $\mathscr{E} \approx a$, to a limiting value $\sigma_\infty$ ($\approx 0.75 \cdot 10^{-38}$ cm$^2$, if $a^2 \approx 37.5\mu_\pi^2$).

The new data on the proton form factors [3] show that while the magnetic form factor falls to zero more rapidly than expected from eq. (5), the charge form factor levels off to a plateau value of $\approx 0.42$. On the basis of eq. (2)-(4) we expect a similar behaviour to be exhibited also by the form factors $F_1$, $F_2$, $H_1$.

This will produce a substantial modification of the quantitative results of I. In fact in the processes we are considering the decrease of the differential cross-section with increasing momentum transfer is only due to the decrease of the form factors of the nucleon. In the case of the scattering of electrons, on the contrary, high momentum transfers are already depressed by the propagator of a virtual photon, which introduces a factor $(k^2)^{-2}$ in the cross section.

**2.** – The differential cross section in the laboratory system for production of e$^\pm$ through processes (1) is given by (I, (7), (8)). If one uses (2)-(4), it can be written:

$$\begin{aligned}\frac{\mathrm{d}\sigma^\pm}{\mathrm{d}\cos\theta} = \frac{G^2\mathscr{E}^2}{\pi}\left(1 + \frac{2\mathscr{E}}{M}\sin^2\frac{\theta}{2}\right)^3 \Bigg\{ & \cos^2\frac{\theta}{2}\left(F_1^V(k^2)\right)^2 + \\ & + \frac{k^2}{4M^2}\left[2\left(F_1^V(k^2) + \mu F_2^V(k^2)\right)^2 \operatorname{tg}^2\frac{\theta}{2} + \mu^2\left(F_2^V(k^2)\right)^2\right]\Bigg\} + \\ & + \left(\frac{G_A}{G}F_1^V(k^2)\right)^2\left[1 + \sin^2\frac{\theta}{2} + 2\frac{\mathscr{E}^2}{M^2}\frac{\sin^4\theta/2}{1 + 2(\mathscr{E}/M)\sin^2\theta/2}\right] \mp \\ & \mp \frac{G_A}{G}F_1^V(k^2)\left(F_1^V(k^2) + \mu F_2^V(k^2)\right)\frac{4\sin^2\theta/2}{1 + 2(\mathscr{E}/M)\sin^2\theta/2}\left(\frac{\mathscr{E}}{M} + \frac{\mathscr{E}^2}{M^2}\sin^2\frac{\theta}{2}\right)\Bigg\}.\end{aligned} \tag{6}$$

Where $\mathscr{E}$ is the neutrino energy, $\theta$ the angle of emission of the electron, $M$ the nucleon mass. We have tabulated this expression for different neutrino energies as a function of $\cos\theta$, making use of two different sets of values for the form factors $F^V$ [4]:

1) *Set* $\alpha$: eq. (5) was used with $a^2 = 37.5\mu_\pi^2$;

2) *Set* $\beta$: the $F^V$ were assumed to coincide with the form factors of the proton as given in [3]; for high momentum transfer $F_1^V$ was assumed to remain at a constant value.

---

[3] R. Hofstadter, F. Bumiller and M. Croissiaux; *Phys. Rev. Lett.*, **5**, 263 (1960).
[4] For both sets, $G_A/G = 1.21$; $\mu = 3.7$.

In Fig. 1 we show the differential cross-section at $\mathscr{E}=1.84M$. Set $\beta$ has a substantial tail at large angles. Table I gives the total cross-section at various energies; set $\beta$ gives higher cross-section, which increase linearly with the energy $\mathscr{E}$.

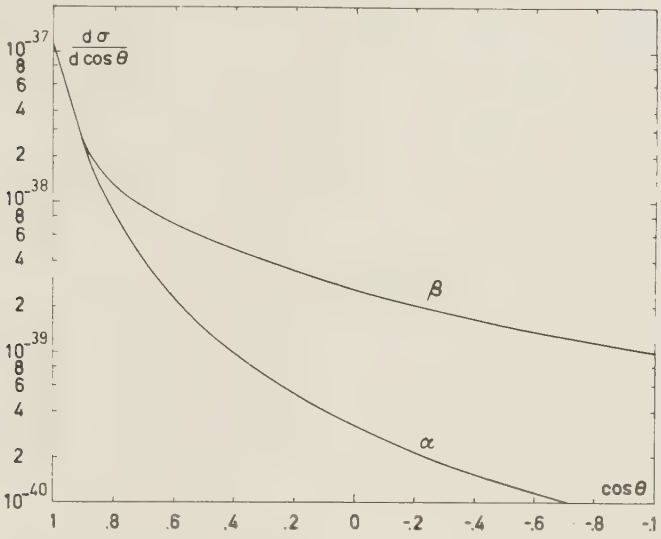


Fig. 1. - Differential cross-section in the laboratory system for the process $\nu+n\to p+e^-$ at a neutrino energy $\mathscr{E}=1.84\ M$.

TABLE I. – *Total cross-section for processes* (1) *at different neutrino energies, in* cm².

| $\mathscr{E}/M$ | Set $\alpha$ | | Set $\beta$ | |
|---|---|---|---|---|
| | $\sigma_\nu \cdot 10^{38}$ | $\sigma_{\bar\nu} \cdot 10^{38}$ | $\sigma_\nu \cdot 10^{38}$ | $\sigma_{\bar\nu} \cdot 10^{38}$ |
| 0.43 | 0.60 | 0.15 | 0.60 | 0.15 |
| 0.62 | 0.75 | 0.22 | 0.75 | 0.22 |
| 0.89 | 0.925 | 0.29 | 0.865 | 0.31 |
| 1.28 | 0.845 | 0.38 | 1.024 | 0.44 |
| 1.84 | 0.832 | 0.46 | 1.31 | 0.60 |
| 2.66 | 0.81 | 0.53 | 1.68 | 0.81 |
| 3.83 | 0.79 | 0.56 | 2.48 | 0.97 |
| 5.52 | 0.76 | 0.59 | 3.69 | 1.72 |

We must remark that set $\beta$ implies an extrapolation of the present knowledge of the nucleon structure to regions of higher momentum transfers, which is perhaps unrealistic, being equivalent to the assumption of a pointlike nucleon core. A bigger uncertainity is due to the ignorance of the neutron form factor. These results, however, stress the fact that the behaviour of the cross-section for processes (1) is strongly dependent on the nucleon form factors for high momentum transfer, and that conversely from these processes we can gain information on features of the nucleon, such as the radius of its core, which are as yet unknown.

* * *

We thank dr. Maria PETILLI, who helped in the numerical calculations.

# Azimuthal Symmetry in 27 GeV Jets.

C. Castagnoli (*), C. Lamborizio and I. Ortalli
*Istituto di Fisica dell'Università - Parma*

A. Barbaro-Galtieri
*Istituto di Fisica dell'Università - Roma*

(ricevuto il 3 Marzo 1961)

## 1. – Introduction.

It has been pointed out by Koba-Takagi [1] Kraushaar-Marks [2] and Ezawa [3], that both in the two center model and in the isobar model of multiple particle production at high energies, an azimuthal asymmetry could be expected. Both models make the hypothesis of a metastable intermediate state in the reaction. It is assumed that high angular momentum partial waves are effective in the collision process, and a larger part of this angular momentum is assigned to the spin of the isobars or fire balls.

If $\overline{p}^*$ is the linear momentum of the isobar and $\overline{p}$ is that of the incident particle, the spin of the isobar will be quantized in the direction of $\overline{p}^* \cdot \overline{p}$. Hence the emission in the $\overline{p}^* \cdot \overline{p}$ plane will be favored when the perpendicolar component of the isobar happens to be large.

Naturally if we find an azimuthal asymmetry we can conclude that the models with intermediate metastable states is rather plausible, while the reverse argument does not hold.

The experimental research of such azimuthal asymmetry has been carried out by some authors [4,6] in cosmic ray jets. We try in the present work to verify the existence or non-existence of such asymmetry by the study of the jets produced by $\sim 27$ GeV protons of the CERN proton-synchrotron.

## 2. – Experimental methods.

An azimuthal asymmetry may be detected through the study of the dependence of the particle density $\mathrm{d}N/\mathrm{d}\Phi$ from the azimuthal angle $\Phi$.

(*) Now at the Istituto di Fisica, Università di Torino.

(1) Z. Koba and S. Takagi: *Nuovo Cimento*, **10**, 755 (1958); S. Hayakawa, Z. Koba and S. Takagi: *Proc. Moscow Conf.*, **1**, 271 (1960).

(2) W. L. Kraushaar and L. J. Marks: *Phys. Rev.*, **83**, 326 (1954).

(3) H. Ezawa: *Nuovo Cimento*, **11**, 745 (1959).

(4) J. Pernegr, V. Petrilka and V. Simak: *Proc. Moscow Conf.*, **1**, 127 (1960).

(5) A. P. Zhdanov, I. M. Kiks, N. V. Skirda and R. M. Yakowlev: *Proc. Moscow Conf.*, **1**, 95 (1960).

(6) E. Balea, E. Friedländer and C. Potocfanu: *Stud. Cercetari Fiz.*, **10**, 413 (1959).

Suppose a plane, normal to the primary, which intersects the jet. Each point of intersection of a jet secondary with the plane is identified be means of co-ordinates $(r, \Phi)$ where $r$ is the distance from the primary intersection, and $\Phi$ the angle with respect to any given direction on the plane. Considering only the variable $\Phi$, the problem of determining an eventual asymmetry is that of determining [7] the probability $P_n(\varphi)$ that, over a circumference, $n$ randomly distributed points form an interval of length between $\varphi$ and $\varphi + \mathrm{d}\varphi$. Comparison $P_n(\varphi)$ with the experimental distribution of $\varphi$ may be carried out by means of the $\chi^2$-test.

It is known that, given an $n$-prong star, the probability that any one of the $n$ azimuthal angles formed by the successive radii dividing up the complete 360°, be not smaller that $\varphi$ radiants $(0 \leqslant \varphi \leqslant 2\pi)$ is given by

$$P_n(\varphi) = \left(1 - \frac{\varphi}{2\pi}\right)^{n-1}. \tag{1}$$

Hence the probability that the angle have any value between $\varphi$ and $\varphi + \mathrm{d}\varphi$ is

$$p_n(\varphi)\,\mathrm{d}\varphi = -\frac{\mathrm{d}P_n(\varphi)}{\mathrm{d}\varphi}\,\mathrm{d}\varphi = \tag{2}$$

$$= (n-1)\left(1 - \frac{\varphi}{2\pi}\right)^{n-2}\frac{\mathrm{d}\varphi}{2\pi}.$$

With this method it is possible to put together all jets with multiplicity $n$ and compare them with $p_n(\varphi)$, with remarkable statistical advantage with respect to other criteria which may be adopted to detect the asymmetry.

We have investigated the distorsion effect on the azimuthal angles $\Phi$ [8], although in our emulsions the distorsion vector $\boldsymbol{D}$ was not very big. The calculation is performed in the two extreme cases of $\boldsymbol{D}$ parallel to the primary directions and of $\boldsymbol{D}$ perpendicular to it.

*a*) In the first case one finds:

$$\operatorname{ctg} \Phi_c = \sin \alpha_c \cdot \operatorname{ctg} \beta_c = \tag{3}$$

$$= \sin \alpha_d \cdot \operatorname{ctg} \beta_d,$$

where $\alpha$ and $\beta$ are the projection and dip angles respectively; the subscripts $c$ and $d$ refer to corrected and distorted angles.

For this expression we can see that the azimuthal angle is not altered by distorsion.

*b*) In the case of orthogonality one finds:

$$\operatorname{ctg} \Phi_d = \sin \alpha_c \operatorname{ctg} \beta_c + A\Delta = \tag{4}$$

$$= \operatorname{ctg} \Phi_c \pm A\Delta,$$

where $A$ is proportional to the distorsion vector and $\Delta$ is proportional to the track distance from the emulsions surface. The calculation has been performed with the maximum value for $A\Delta$ (mean value on the total thickness) in our emulsions ($A\Delta = 0.17$).

We have found that the distorsion effect amounts at a few percent only for large $\varphi$, which have small weight in the distribution (2) [for $\varphi = 100°$ is $p(\varphi_c) = 10.8 \cdot 10^{-4}$ and $p(\varphi_d) = 11.0 \cdot 10^{-4}$].

We conclude that the distortion vector, although not negligible, does not alter the parameter $\varphi$, whatever is the jet divergence, owing to the fact that (4) is independent from $\alpha$ and $\beta$.

## 3. – Results.

We have measured the angular distribution of 142 jets with $n_h = 0.1$ and $2 \leqslant n_h \leqslant 4$. The former may be attri-

(7) D. P. STERN: *Suppl. Nuovo Cimento*, **10**, 251 (1960).

(8) C. CASTAGNOLI and A. MANFREDINI: *Sur l'influence de la distorsion dans les distributions angulaires*, in DEMERS: *Photographie corpuscolaire*, vol. 2, (Montreal, 1959), pp. 276-280.

buted, to a certain extent, to nucleon-nucleon collisions, while the latter are for the most attributed to collisions against the light nuclei, C, N, O. The value of the projected angle $\alpha$ and of the dip angle $\beta$, were plotted on a Wulff diagram so as to determine the corresponding $\varphi$ with respect to any given prong.

shown in Table I. As is seen, for the multiplicities considered, the $\chi^2$-method indicates that, with a high degree of reliability, the $\varphi$ distribution here considered agrees with the one to be expected under the hypotesis of azimuthal symmetry.

This result is confirmed by the

TABLE I.

| $n_s$ | $n_h$ | frequency | | | | | | | | $\chi^2$ | $P(\chi^2)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 6 | 0, 1 | observed | 53 | 42 | 38 | 32 | 20 | 13 | 11 | 11 | 8.0 | 0.24 |
| | | calculated | 52.5 | 43.1 | 35.2 | 29.4 | 23.8 | 18.0 | 12.0 | 6.0 | | |
| 7 | 0, 1 | observed | 34 | 31 | 19 | 18 | 9 | 7 | 7 | 3 | 9.3 | 0.16 |
| | | calculated | 40 | 25.9 | 21 | 15.1 | 10.5 | 5.8 | 3.0 | 1.7 | | |

The values of $\varphi$ were thus collected, jet by jet, corresponding to the angles between successive prongs; and single distributions of $\varphi$ were obtained, corresponding to jets of given multiplicities $n_s$.

The $\chi^2$-method of analysis was used for the various multiplicities comparing (2) with the corresponding experimental distribution. As examples, the results for $n_s = 6, 7$ and $n_h = 0, 1$ are

values of $P(\chi^2)$ which are comprised between 0.10 and 0.70 for other multiplicities ($8 \leqslant n_s \leqslant 13$). With our statistics at this energy the differing behaviour at high and low multiplicities which is observed [5] in cosmic ray jets is not detected.

* * *

We wish to thank the PS Machine and Emulsion Groups of CERN for exposure of the stack.

IL NUOVO CIMENTO Vol. XX, N. 2 16 Aprile 1961

# Vector Field Associated with the Unitary Theory of the Sakata Model.

A. Salam

*Imperial College - London*

J. C. Ward

*Carnegie Institute of Technology - Pittsburgh*

(ricevuto il 13 Marzo 1961)

The possibility that weak and strong interactions might be mediated by vector fields associated with gauge transformations has come to be more and more discussed in recent theoretical work [1]. In an earlier paper [2] the authors proposed the view that if the set of « elementary » particles and their symmetry properties were known, one could generate their elementary interactions unambiguously by replacing ordinary derivatives in the free Lagrangian by « co-variant derivatives ». These co-variant derivatives are the « gauge-invariant derivatives » defined in the internal symmetry space to which the particles may belong.

For the Sakata set of elementary particles (p, n, $\Lambda$) it has been suggested that the internal symmetry space is a three-dimensional unitary space [3]. More concretely if $\chi = \begin{pmatrix} \mathrm{p} \\ \mathrm{n} \\ \Lambda \end{pmatrix}$, the kinetic energy part of the Lagrangian

$$\chi^{+}\gamma_4\gamma_\mu \frac{\partial}{\partial x_\mu}\chi\,, \tag{1}$$

is invariant for the unitary transformation

$$\chi \to (1 + iH)\chi\,, \tag{2}$$

where $H$ is an 8-parameter, traceless (infinitimal) Hermitian matrix. (The mass-terms in the Lagrangian are not invariant since $m_N \neq m_\Lambda$). If $H$ varies over space-

---

(1) C. N. Yang and R. Mills: *Phys. Rev.*, **96**, 191 (1954); R. Shaw: *Cambridge Dissertation* (1954), unpublished; A. Salam and J. C. Ward: *Nuovo Cimento*, **11**, 568 (1959); J. J. Sakurai: *Ann. of Phys.*, **11**, 1 (1960).

(2) A. Salam and J. C. Ward: *Nuovo Cimento*, **19**, 167 (1961).

(3) M. Ikeda, S. Ogawa and Y. Ohnuki: *Progr. Theor. Phys.*, **22**, 715 (1959); Y. Yamaguchi: *Progr. Theor. Phys. Suppl.*, no. 11 (1959); J. Wess: *Nuovo Cimento*, **10**, 15 (1960).

time, (1) is invariant only if $\partial/\partial x_\mu$ is replaced by the co-variant derivative

$$\nabla_\mu = \frac{\partial}{\partial x_\mu} - iH_\mu , \tag{3}$$

where $H_\mu$ is a traceless Hermitian matrix with the elements

$$H_\mu = \begin{pmatrix} \frac{1}{\sqrt{6}}\varrho^0_\mu + \frac{1}{\sqrt{2}}\pi^0_\mu & \pi^+_\mu & K^+_\mu \\ \pi^-_\mu & \frac{1}{\sqrt{6}}\varrho^0_\mu - \frac{1}{\sqrt{2}}\pi^0_\mu & K^0_\mu \\ K^-_\mu & \bar{K}^0_\mu & -\frac{2}{\sqrt{6}}\varrho^0_\mu \end{pmatrix} . \tag{4}$$

Here $\pi_\mu$ and $K_\mu$ are unit-spin fields with the same symmetry properties ($i$-spin, strangeness, etc.) as $\pi$- and K-mesons and $\varrho^0_\mu$ is a neutral ($I=0$) particle. To show this the simplest procedure is to remark that so far as symmetry properties are concerned, the simplest representation of the Lie-matrix $H$ is generated by

$$H = \chi\chi^+ = \begin{pmatrix} \mathrm{p} \\ \mathrm{n} \\ \Lambda \end{pmatrix} (\mathrm{p}^+ \mathrm{n}^+ \Lambda^+) \tag{5}$$

$$\begin{pmatrix} \mathrm{pp}^+ & \mathrm{pn}^+ & \mathrm{p}\Lambda^+ \\ \mathrm{np}^+ & \mathrm{nn}^+ & \mathrm{n}\Lambda^+ \\ \Lambda\mathrm{p}^+ & \Lambda\mathrm{n}^+ & \Lambda\Lambda^+ \end{pmatrix} . \tag{6}$$

This matrix is not traceless and therefore does not generate an irreducible Algebra. Subtracting the trace element, $\frac{1}{3}(\bar{p}p+\bar{n}n+\bar{\Lambda}\Lambda)$ and identifying

$$\pi^0 = \frac{1}{\sqrt{2}}(\bar{p}p - \bar{n}n) , \qquad \varrho^0 = \frac{1}{\sqrt{6}}(\bar{p}p + \bar{n}n - 2\bar{\Lambda}\Lambda) ,$$

we obtain the symmetry identifications (4).

The full Sakata Lagrangian now reads:

$$\chi^+\gamma_4\gamma_\mu \frac{\partial}{\partial x_\mu}\chi - i\chi^+\gamma_4\gamma_\mu H_\mu \chi + \frac{\mathrm{Tr}}{3}(\nabla_\mu H_\nu - \nabla_\nu H_\mu)^2 + \tag{7}$$

$$+ \text{ mass terms for p, n, } \Lambda \text{ and H-fields} .$$

Thus in addition to the three original particles, the theory would contain eight vector mesons: four ($K_\mu$) resembling K-mesons in their isotopic character, three ($\pi_\mu$) resembling pions and one particle ($\varrho^0_\mu$), an isoscalar. The $K_\mu$ could decay into $K+\pi$, while $\varrho_\mu$ might decay into $3\pi$. The three particle $\pi_\mu$ are identical with the triplet $B_T$ introduced by SAKURAI (*).

(*) C. N. CHAN (*Phys. Rev. Lett.*, to be published) has shown from the relation between decay-time and production probability for the newly discovered K* particle (reported by ALSTON *et al.* at the Berkeley Conference on Strong Interactions) that it must carry unit spin. One wonders if it may be identified with $K_\mu$. Similarly the ABASHIAN, BOOTH and CROWE particle also reported at this Conference may be the $\varrho^0_\mu$. This was speculated on by J. J. SAKURAI (Berkeley Conference on Strong Interactions) and M. GELL-MANN.

It is possible to introduce pseudo-vector mesons in addition to vector mesons, if we consider the gauges

$$\chi_L \to (1 + iH_1)\chi_L\,, \qquad \chi_R \to (1 + iH_2)\chi_R\,,$$

where

$$\chi_{L,R} = \tfrac{1}{2}(1 \pm \gamma_5)\chi\,.$$

The combinations $H_\mu = \frac{1}{2}(H_{1\mu} + H_{2\mu})$ and $H' = \frac{1}{2}(H_{1\mu} - H_{2\mu})$ (*) are vector and pseudo-vector respectively.

Similar ideas seem to provide the nearest approximation to a gauge theory of weak vector bosons. An irreducible realization of the unitary Lie algebra is provided by the set of matrices

$$B = \begin{pmatrix} -\sqrt{2}\,b^0 & b & b \\ b^* & \frac{1}{\sqrt{2}}\,b^0 & \frac{1}{\sqrt{2}}\,b^0 \\ b^* & \frac{1}{\sqrt{2}}\,b^0 & \frac{1}{\sqrt{2}}\,b^0 \end{pmatrix}.$$

where $b^0$ is real.

A gauge transformation $\chi_L \to (1 + iB)\chi_L$ could lead to a weak interaction of the type

$$(\bar{p}_L\gamma_\mu n_L + \bar{p}_L\gamma_\mu \Lambda_L)B_\mu^+ + \frac{1}{\sqrt{2}}\,(-2\bar{p}_L\gamma_\mu p_L + \bar{n}_L\gamma_\mu n_L + \bar{\Lambda}_L\gamma_\mu\Lambda_L + \bar{n}_L\gamma_\mu\Lambda_L + \bar{\Lambda}_L\gamma_\mu n_L)\,B_\mu^0\,.$$

Note the natural way in which the charged intermediate boson $B_\mu^+$ for weak interactions makes its appearance. This interactions however is likely to lead to transitions involving $\Delta S = 2$. The triplet consisting of $\begin{pmatrix} \nu \\ e \\ \mu \end{pmatrix}$ for leptons would have obvious analogies with $\begin{pmatrix} p \\ n \\ \Lambda \end{pmatrix}$.

***Note added in proof.***

After the work described above was completed, the authors learnt of very similar ideas by Y. Neeman and M. Gell-Mann.

(*) The presence of the non-invariant baryon mass terms in the Sakata Lagrangian means the divergence of the pseudo-vector current does not vanish. It is an interesting possibility that the components of $\partial H_\mu/\partial x'_\mu$ are the $\pi$ and the K fields themselves.

# A Solvable Model of Field Theory (*).

E. R. Caianiello and A. Campolattaro

*Istituto di Fisica Teorica dell'Università - Napoli*
*Scuola di Perfezionamento in Fisica Teorica e Nucleare - Napoli*

(ricevuto il 27 Marzo 1961)

**1.** – Consider a Dirac spinor field coupled scalarly with a neutral boson field; the propagation kernels (1) that correspond to $N$ incoming, $N$ outgoing spinor lines and no external boson lines have then the perturbative expansions:

$$(1) \qquad K\begin{pmatrix} x_1 \dots x_N \\ y_1 \dots y_N \end{pmatrix} = \sum_{n=0}^{\infty} \left(\frac{\lambda^2}{2}\right)^n \frac{1}{n!} \int \mathrm{d}\xi_1 \dots \int \mathrm{d}\xi_{2n} \begin{pmatrix} x_1 \dots x_N \ \xi_1 \dots \xi_{2n} \\ y_1 \dots y_N \ \xi_1 \dots \xi_{2n} \end{pmatrix} [\xi_1 \xi_2] \dots [\xi_{2n-1} \xi_{2n}] .$$

and satisfy the eqs. (2)

$$(2) \qquad \frac{\partial}{\partial(\lambda^2/2)} K\begin{pmatrix} x_1 \dots x_N \\ y_1 \dots y_N \end{pmatrix} = \int \mathrm{d}\xi_1 \int \mathrm{d}\xi_2 [\xi_1 \xi_2] K\begin{pmatrix} \xi_1 \xi_2 \ x_1 \dots x_N \\ \xi_1 \xi_2 \ y_1 \dots y_N \end{pmatrix},$$

$$(3) \qquad \frac{\partial}{\partial m} K\begin{pmatrix} x_1 \dots x_N \\ y_1 \dots y_N \end{pmatrix} = -i \int \mathrm{d}\xi K\begin{pmatrix} \xi \ x_1 \dots x_N \\ \xi \ y_1 \dots y_N \end{pmatrix},$$

where $(xy)$ and $[xy]$ are the free Feynman propagators for the spinor and the boson field, $\lambda$ is the coupling constant, $m$ is the fermion mass, and all other notations are as in ref. (1) (cf. formula (58)) and ref. (2) (cf. formulae (23-24)).

Integrations in the above equations are understood to be over all space and time for the sake of simplicity; it is relevant to remark, for future reference, that finite intervals of integration can be handled by this formalism with equal ease.

We obtain our model by replacing throughout the free boson propagator with a positive constant, which without loss of generality we may take equal to one. The model describes thus a non-local « phonon » interaction among spinor particles, such that each point of a given spinor line interacts equally with all past, present and future points of the same and all other lines; the vacuum fluctuations of the spinor field are taken into account correctly.

(*) Research reported in this document has been sponsored in part by the European Office of the A.R.D.C., United States Air Force, with Contract no. AF 61(052)434.

(1) E. R. Caianiello: *Nuovo Cimento*, **11**, 492 (1954).

(2) E. R. Caianiello: *Nuovo Cimento*, **13**, 637 (1959).

We have thus, from (2) and the iteration of (3):

$$\frac{\partial}{\partial(\lambda^2/2)} K\begin{pmatrix} x_1 \dots x_N \\ y_1 \dots y_N \end{pmatrix} + \frac{\partial^2}{\partial m^2} K\begin{pmatrix} x_1 \dots x_N \\ y_1 \dots y_N \end{pmatrix} = 0 , \tag{4}$$

where, from now on, $K\begin{pmatrix} x_1 \dots x_N \\ y_1 \dots y_N \end{pmatrix}$ denote the kernels relative to our model.

(4) is obviously solved formally by the expression

$$K\begin{pmatrix} x_1 \dots x_N \\ y_1 \dots y_N \end{pmatrix} = \exp\left[-\frac{\lambda^2}{2}\frac{\partial^2}{\partial m^2}\right]\begin{pmatrix} x_1 \dots x_N \\ y_1 \dots y_N \end{pmatrix}, \tag{5}$$

where at r.h.s. we find the symbolic product of an operator that contains the interaction, or « dynamics », times a determinant expressing the free propagation of $N$ fermions, or « kinematics », of the system. From the standard theory of parabolic equations it is immediate to see that (5) is the only permissible formal solution which reduces to the free propagator when $\lambda=0$.

This might conclude our discussion, because the solution of parabolic equations is a classical chapter of analysis.

It is instructive, however, to discuss (4) and (5) more thoroughly, first with the customary heuristic methods and then with greater mathematical care; we shall be able thus to see clearly how the infinities that arise in the first attempt are consistently removed by an appropriate handling of limiting processes.

**2.** – It will suffice to consider the case $N=1$; the same procedure would apply as well, but for formal complications, for $N>1$. We proceed at first heuristically by assuming that (5) is meaningful as it is and disregarding questions of commutability when infinite processes are performed. We transform thus $K\begin{pmatrix} x \\ y \end{pmatrix}$ into its expression in momentum space:

$$K(p) = \exp\left[-\frac{\lambda^2}{2}\frac{\partial^2}{\partial m^2}\right]\frac{1}{i\gamma p + m}, \qquad (m \equiv m - i\varepsilon); \tag{6}$$

the r.h.s. of (6) becomes, if the exponential operator is given its usual meaning:

$$\frac{1}{i\gamma p + m}\sum_{n=0}^{\infty}\frac{(2n)!}{n!}\left(-\frac{\lambda^2}{2}\frac{1}{(i\gamma p + m)^2}\right)^n, \tag{7}$$

which has radius of convergence zero and is therefore a divergent quantity.

The only legitimate conclusion, of course, is that the expansion (7) is incorrect if the exponential operator is applied to $1/(i\gamma p+m)$; the r.h.s. of (6) may at best be regarded as a purely symbolic notation, which has yet to be specified completely, if this is possible.

Take now:

$$\frac{1}{i\gamma p + m} = \int_0^\infty \exp\left[-(i\gamma p + m)t\right] dt , \tag{8}$$

substitute (8) into (6) and bring *arbitrarily* the exponential operator within the integral sign: then

$$(9) \qquad K(p) = \int_0^\infty \exp\left[-\frac{\lambda^2}{2} t^2\right] \exp\left[-(i\gamma p + m)t\right] \mathrm{d}t = \frac{1}{\sqrt{2}\lambda} \Psi\left(\frac{1}{2}, \frac{1}{2}, \left[\frac{i\gamma p + m}{\sqrt{2}\lambda}\right]^2\right),$$

where $\Psi(a, b, x)$ is Tricomi's confluent hypergeometric function of parameters $a$, $b$ and argument $x$ (3).

It is then immediate to verify *a*) that $K(p)$, as given by (9), satisfies the eq. (4) and *b*) that

$$(10) \qquad \lim_{\lambda \to 0} \frac{1}{\sqrt{2}\lambda} \Psi\left(\frac{1}{2}, \frac{1}{2}, \left[\frac{i\gamma p + m}{\sqrt{2}\lambda}\right]^2\right) = \frac{1}{i\gamma p + m},$$

since (setting $z = (i\gamma p + m)/\sqrt{2}\lambda$),

$$(11) \qquad \Psi\left(\frac{1}{2}, \frac{1}{2}, z^2\right) = 2 \exp[z^2] \int_z^\infty \exp[-t^2] \mathrm{d}t = z\Psi\left(1, \frac{3}{2}, z^2\right),$$

and

$$(12) \qquad \lim_{\xi \to \infty} \xi \Psi(1, \tfrac{3}{2}, \xi) = 1.$$

We find thus *a posteriori* that the expression (9) of $K(p)$ is indeed the correct solution of our problem.

**3.** – From a mathematical point of view this is all we can ask for; also the question of the definition of the symbol $\exp[-(\lambda^2/2)(\partial^2/\partial m^2)]$ receives a precise answer, because we may, *e.g.*, prescribe that it operates only according to the sequence of steps shown above: the first faulty step was careless passage to momentum space in (6). We wish however to investigate further this situation, which is otherwise perfectly clear, so as to bring into light the connection between the « wrong approach » (7), the « correct approach » (9) and « renormalization ».

We wish first to note the following relation:

$$(13) \qquad \exp\left[\frac{1}{4}\left(1 - \frac{1}{\mu}\right)\frac{\mathrm{d}^2}{\mathrm{d}z^2}\right] \Psi\left(\frac{1}{2}, \frac{1}{2}, z^2\right) = \sqrt{\mu}\, \Psi\left(\frac{1}{2}, \frac{1}{2}, \mu z^2\right);$$

the exponential operator at l.h.s. of (13) is understood, as in (7), to denote the power series expansion. (13) is a straightforward consequence of the multiplication theorem, taken for $a = 1$, $b = \frac{3}{2}$, $x = z^2$

$$(14) \qquad \Psi(a, b, \mu x) = \mu^{-a} \sum_{n=0}^\infty \frac{\Gamma(a + n)\Gamma(a - b + 1 + n)}{\Gamma(a)\Gamma(a - b + 1)n!} \left(1 - \frac{1}{\mu}\right)^n \Psi(a + n, b, x),$$

(3) F. G. Tricomi: *Funzioni ipergeometriche confluenti* (Roma, 1954).

(valid for $|\mu-1|<1$, $\mathrm{Re}\,\mu>\frac{1}{2}$) and for the relation (which essentially defines at l.h.s. a class of parabolic cylinder functions):

$$z\Psi\left(n+1,\frac{3}{2},z^2\right)=\frac{1}{(2n)!}\frac{\mathrm{d}^{2n}}{\mathrm{d}z^{2n}}\Psi\left(\frac{1}{2},\frac{1}{2},z^2\right). \tag{15}$$

We may now replace (6) with the *regularized expression*:

$$K_{\alpha,\lambda}(p)=\exp\left[-\frac{\lambda^2}{2}\frac{\partial^2}{\partial m^2}\right]P_{\alpha,\lambda}(p)\,, \tag{16}$$

where:

$$P_{\alpha,\lambda}(p)=\frac{1}{\sqrt{2}\lambda}\frac{\alpha}{\sqrt{1-\alpha^2}}\Psi\left(\frac{1}{2},\frac{1}{2},\frac{\alpha^2}{1-\alpha^2}\left[\frac{i\gamma p+m}{\sqrt{2}\lambda}\right]^2\right), \tag{17}$$

and

$$K_{\alpha,\lambda}(p)=\frac{1}{\sqrt{2}\lambda}\alpha\Psi\left(\frac{1}{2},\frac{1}{2},\alpha^2\left[\frac{i\gamma p+m}{\sqrt{2}\lambda}\right]^2\right); \tag{18}$$

the equality (16) follows from (13) with

$$z=\frac{\alpha}{\sqrt{1-\alpha^2}}\frac{i\gamma p+m}{\sqrt{2}\lambda}\,,\qquad \mu=1-\alpha^2\,,$$

and is certainly valid if $|\alpha^2|<1$, $\mathrm{Re}\,\alpha^2<\frac{1}{2}$.

We observe that, as in (10), (11), (12),

$$\lim_{\alpha\to 1}P_{\alpha,\lambda}(p)=\frac{1}{i\gamma p+m}\,; \tag{19}$$

therefore, if we take in (16) *first* the limit $\alpha\to 1$ and *then* apply the exponential operator at r.h.s., the r.h.s. reduces to the divergent expression (7); if, instead, we *first* apply the exponential operator and *then* take the limit $\alpha\to 1$, we obtain the correct solution (9). Taking this limit is clearly a process of *continuation*, of the same nature as those which are met in the study of renormalization with finite-part integrals.

This shows how things depend critically upon the order in which infinite processes are taken. We note furthermore that, since it is formally evident from (18) that

$$K_{\alpha,\lambda}(p)=K_{\alpha',(\lambda\alpha'/\alpha)}(p)=K_{1,(\lambda/\alpha)}(p)\,, \tag{20}$$

the transition from a value to another of the index $\alpha$ could be interpreted as a « charge renormalization »; this would be, in the customary language, the « physical meaning » of the transition from (7) to (9).

We forego commenting here upon the many questions which naturally arise at this point; unitarity, causality, existence of a renormalization group (heuristically, the theory is evidently hyper-renormalizable). The answers would be straightforward, but there is too little physics in this model to warrant the effort; the interest of the present considerations lies rather, in our opinion, in the fact that it seems possible to extend them to more realistic situations, *e.g.* in the physics of many-body problems, where they may yield novel methods of approximation.

# LIBRI RICEVUTI E RECENSIONI

D. Blanc et G. Ambrosino – *Eléments de physique nucléaire*. Pp. IX-238 con 94 figg., Masson & Cie, Editeurs, 30 N.F.

Già da diversi anni esistono libri di fisica nucleare a carattere introduttivo, scritti allo scopo di costituire un punto di partenza (in alcuni casi un punto di arrivo) per tutti coloro che, in possesso delle nozioni di fisica e di matematica impartite nel primo biennio dei corsi di laurea in fisica, ingegneria, matematica e chimica, desiderino avere un quadro approssimativamente completo ed aggiornato della situazione odierna della fisica nucleare. Questi testi sono particolarmente dedicati agli ingegneri (o chimici) che hanno seguito corsi di laurea tradizionali e che tendono a trasformarsi in specialisti nucleari.

A questo tipo appartiene il libro di D. Blanc e G. Ambrosino. L'esposizione è brillante e molto scorrevole. In qualche capitolo gli argomenti sono trattati forse troppo brevemente. Le proprietà dei neutroni lenti, per esempio, sono descritte senza accennare al loro impiego nella ricerca. Queste lacune sono però in parte giustificabili se si considera la finalità del libro, il quale è corredato da copiosi grafici e tabelle che risultano molto efficaci per la comprensione degli argomenti trattati.

Gli autori seguono l'impostazione oramai tradizionalmente adottata per la scelta e la successione degli argomenti trattati. Il libro si articola in nove capitoli, un'appendice ed indici per materie e per autori. I primi due capitoli sono a carattere introduttivo, il terzo capitolo tratta i nuclei, il quarto le disintegrazioni radioattive, il quinto le reazioni nucleari. Il sesto è dedicato ai neutroni mentre il settimo tratta la fissione e le reazioni a catena. Il capitolo ottavo è dedicato ai modelli nucleari. L'ultimo capitolo tratta infine le reazioni termonucleari e descrive alcune apparecchiature costruite per lo studio della fusione. Ogni capitolo è dotato di un'ampia ed aggiornata bibliografia.

In conclusione si tratta di un buon libro che può senz'altro inserirsi nell'elenco dei testi introduttivi consigliati.

S. Sciuti

R. C. Koch – *Activation Analysis Handbook*. Academic Press, New York-London, 1960, pp. 219-645; s. 64.

Delle 219 pagine di cui è costituito il manuale, 200 sono di tabelle.

Le prime 19 pagine trattano in modo sommario le considerazioni teoriche riguardanti l'analisi per attivazione, con ampio riferimento alla bibliografia.

Le citazioni bibliografiche sommano a 570 di cui 503 riguardanti lavori pubblicati fino all'Ottobre 1958 e le rimanenti 67 al Settembre 1959.

Le tabelle raccolgono i dati necessari nei problemi di analisi per attivazione per gli elementi dall'idrogeno al plutonio.

Per ogni elemento vengono riportati i dati relativi all'attivazione con neutroni e con particelle cariche o con fotoni.

In particolare per l'attivazione con neutroni sono elencati:

*a*) l'isotopo con l'abbondanza e la sezione d'urto;

*b*) il tempo di dimezzamento e l'energia delle particelle $\beta$ e $\gamma$ emesse dal radioelemento formato;

*c*) la sezione d'urto delle reazioni di secondo ordine ed il tempo di dimezzamento dei prodotti formati;

*d*) informazioni sull'analisi per attivazione con neutroni lenti riguardo alle reazioni nucleari, matrici, irradiazioni speciali, sensibilità, flussi, reazioni interferenti osservate, commenti speciali;

*e*) impiego di neutroni veloci con le stesse voci che in *d*) più le soglie, le sezioni d'urto o le funzioni di eccitazione;

*f*) dati per le altre reazioni con neutroni, riguardanti le reazioni stesse, le soglie, le sezioni d'urto o le funzioni di eccitazione;

*g*) le possibili reazioni interferenti primarie, secondarie, di secondo ordine e gli effetti di autoschermaggio;

*h*) i microgrammi di elemento rivelabili, con le reazioni riportate, nelle condizioni standard di irradiazione e di misura specificate nelle considerazioni teoriche.

Per l'attivazione con particelle cariche o con fotoni sono elencate:

*a*) le reazioni nucleari, le matrici, le tecniche speciali di irradiazione e di saggio, la sensibilità, l'intensità del fascio, le reazioni interferenti osservate, i commenti speciali, le soglie, le sezioni d'urto o di eccitazione;

*b*) dati per le reazioni con altre particelle cariche o fotoni riguardanti le reazioni stesse, le soglie, le sezioni d'urto o le funzioni di eccitazione;

*c*) le possibili reazioni interferenti primarie, secondarie, di secondo ordine, gli effetti di autoschermaggio;

*d*) i microgrammi di elemento rivelabili per centimetro quadro con le reazioni riportate, nelle condizioni standard di irradiazione e di misura specificate nelle considerazioni teoriche.

Per tutti i dati riportati nelle tabelle è segnalata la fonte bibliografica.

Per quanto sopra esposto è evidente l'utilità del manuale a coloro che si interessano di analisi per attivazione, specialmente per ciò che riguarda la rapidità nel reperire i dati necessari nei problemi di applicazione del metodo.

F. Salvetti



Direttore responsabile: G. POLVANI Tipografia Compositori - Bologna

Questo fascicolo è stato licenziato dai torchi il 26-IV-1961